An aggregate advertising response model based on consumer population dynamics

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Abstract: Aggregate advertising models are functions reflecting the relationship between product sales and advertising expenditure for a market as a whole. In this paper, we proposed a new advertising response model based on consumer population dynamics. Considering that the population dynamics is one of the basic characteristics of consumers, we try to apply it in describing the effects of advertising. In detail, a two-level framework advertising response model is introduced in this paper, in which the general Lotka-Volterra model is used to describe population dynamics among consumers and another increasing function is utilised to reflect the advertisement's effect on the intrinsic sales growth rate of a product or a brand. Mathematic analysis shows that the new advertising model has more advantages than other classic models such as Vidale-Wolfe model, Nerlove-Arrow model, Lanchester model and their modifications. Also, an early-warning marketing mechanism is introduced as an application of the proposed model.

Keywords: consumer population dynamics; advertising response model; Lotka-Volterra model; early-warning marketing mechanism.

Reference to this paper should be made as follows: Wang, M., Gou, Q., Wu, C. and Liang, L. (xxxx) 'An aggregate advertising response model based on consumer population dynamics', *Int. J. Applied Management Science*, Vol. X, No. Y, pp.000–000.

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1 Introduction

Aggregate advertising response model is a function which reflects the relationship between product sales and the advertising spending for a market as a whole. It cannot only be used to describe how the advertising investments influence the market demands of products, but also be utilised to optimise the firm's decisions on the advertising investments, where the decisions include planning, budgeting, forecasting and controlling (Hanssens et al., 2009; Yan and Wang, 2009).

As the foundation of decision models, three kinds of advertising response models are usually studied in previous literatures, including priori models, econometric models and mixed models (Little, 1979). Based heavily on intuition without considering the empirical data, priori models are utilised to postulate a general structure, rather than describe a specific application (Little, 1979). From the empirical data, the econometric model specifies the statistical relationship among the various marketing quantities (e.g., sales, market shares, advertising budgets, entry time, retail prices, etc.) to pertain some particular advertising effects such as advertising threshold effects, brand halo effects, multi-media synergy effects and so on. Integrating the priori model with the econometric model, the mixed model starts with rather more complicated a priori model and endeavours by statistical methods to fit and evaluate them.

The priori advertising models can be classified into two streams, i.e., that based on Vidale-Wolfe model (Vidale and Wolfe, 1957) and that on Nerlove and Arrow model (Nerlove and Arrow, 1962). Based on the informative effect of advertising, the Vidale-Wolfe model and its modifications argue that the advertising can arouse more potential consumers to know the products and exert a direct positive influence on the sales of the firms. The basic idea of Nerlove-Arrow model is that continuous advertising can generate an accumulation of goodwill, and the goodwill usually has positive effects on the sales.

As illustrated by Little (1979), a good priori model should explain or fit most advertising effects which have been proven by the empirical studies. Summarising conclusion of previous empirical studies, Little (1979) introduced five basic criteria which the priori should satisfy (see Section 4). Unfortunately, both the Vidale-Wolfe and Nerlove-Arrow model fail in some of these criteria.

Considering the above facts, this paper introduces a new priori advertising response model based on Lotka-Volterra framework, and proves that the new model can fit all the five criteria. Specifically, we prove that our model has more advantages than other classic advertising models such as Vidale-Wolfe model, Nerlove-Arrow model, Lanchester model and their modifications. Also, an early-warning marketing mechanism aiming that the firm can react to the marketing strategy of its competitor quickly is introduced as an application of the proposed model.

The rest of this paper is organised as follows. Review of related literatures is in Section 2. We develop the model in Section 3 and check whether our model fits the Little's five criteria in Section 4. Analysis of the stable equilibrium of the proposed model and an early warning marketing mechanism are presented in Section 5. Concluding remarks are given in Section 6.

2 Literature review

Related literatures of the paper are mainly from that on aggregate advertising response models and that on Lotka-Volterra models.

Advertising response models usually can be classified into priori models, econometric models and mixed models. Commonly, econometric models could have a linear (e.g., Bass and Clarke, 1972; Bass and Leone, 1983; Greuner et al., 2000; Telang et al., 2004; Song and Mela, 2009) or non-linear form (Lambin, 1972; Clarke, 1973; Vakratsas et al., 2004; Ghose and Yang, 2009; Yang and Ghose, 2010). Integrating the priori model with the econometric model, the mixed model starts with rather more complicated a priori model and endeavours by statistical methods to fit and evaluate them, examples include Kuehn et al. (1966), Rao (1986), Golan et al. (2000), Amaldoss and He (2009), Mesak and Ellis (2009), and Masanell and Zhu (2010).

There are two kinds of models in the priori advertising models. One is Vidale-Wolfe model (Vidale and Wolfe, 1957) and its modifications (Kimball, 1957; Ozga, 1960; Sethi, 1983; Sorger, 1989; Wang and Zhang, 2001; Erickson, 2009; Mesak and Ellis, 2009). Based on the informative effects of advertising, the Vidale-Wolfe model argues that the advertising can arouse more potential consumers to know the products and exert a positive influence on the sales of the firms. Thus, in the Vidale-Wolfe model the sales rate of a product (or its market share) is linear with the product of the advertising level and the potential market size. Modifications for the original Vidale-Wolfe model include the following:

1. to introduce the Lanchester model as an extension to concern the dynamic shifts in the market shares of multiple competitors (Kimball, 1957)
2. to consider that the customers who have already purchased the products can affect the untapped part of the market (Ozga, 1960)
3. to regard that the rate at which the awareness is generated is proportional to the advertising effort and the square root of the unaware customers (Sethi, 1983)
4. to take advertising diminishing marginal utility into account (Sorger, 1989)

5 to add the competitive dynamics into the original model (Wang and Zhang, 2001)

6 to ameliorate the Vidale-Wolfe model to allow dynamic analysis of an oligopoly in which the each competitor offers multiple brands (Erickson, 2009)

7 to study the superiority of advertising pulsing policy (turning advertising on and off in a cyclic fashion) over its uniform (constant spending) counterpart that costs the same under the assumption that sales dynamics follow a modified Vidale-Wolfe aggregate advertising model (Mesak and Ellis, 2009).

The other is Nerlove-Arrow model and its modifications. Different from the Vidale-Wolfe model which believes that advertising has a direct impact on the sales, Nerlove-Arrow model assumes that advertising can accumulate a kind of goodwill and this goodwill will influence the sales. Modifications of the Nerlove-Arrow model include:

1 to distinguish the advertising into long-term and short-term advertising effects and assume that only the long-term advertising effect will affect the goodwill (Jørgensen et al., 2000)

2 to suppose that national advertising has positive impacts on goodwill accumulation whereas local advertising has negative effects (Fu and Zeng, 2007)

3 to consider that advertising efforts of one brand may hurt the competitor's goodwill stock (Amrouche et al., 2008)

4 to extend the linear Nerlove-Arrow model to a non-linear model of advertising theme quality and goodwill and estimate the extended model using Markov chain Monte Carlo and particle filtering ideas (Bruce, 2008).

As a famous model in describing the population dynamics between two biological species, the Lotka-Volterra model was not only widely used in the ecosystem of competing species (Mccarley and Hobson, 1975; Holt and Pickering, 1985; Chesson, 1994; Emmerson and Raffaelli, 2004), but also introduced into the economic field for the study of the dynamics of consumer durable goods (Parker and Gatignon, 1994), securities research (Modis, 1999), population control (Delfino and Simmons, 1999), economic growth (Delfino and Simmons, 2000) and other issues (Farmer, 2000; Slobodyan, 2001; Kong, 2005). For example, based on Lotka-Volterra model, Parker and Gatignon (1994) studied the dynamics of consumer durable goods when a new competitor emerges in the oligopoly market; Modis (1999) discussed the competition relationship between the stock and bond market; Delfino and Simmons (2000) studied the population growth and disease problems based on the Lotka-Volterra system, and then analysed the path of economic growth in the case of savings outside.

3 Model developments

3.1 The consumer's population dynamics

Population dynamics is one of the basic characteristics of consumers. According to different criteria, a consumer can be classified into various consumer groups or categories (Kotler et al., 1999; Wang et al., 2008). For example, based on the age structure, a consumer may belong to young, middle, or old age generation; distinguished from the attitude of accepting a new technology, a consumer may be an early adopter or a follower (Aravamudhan, 2011). Specifically, from the attitude to different brands of a product category, consumers can be divided into different brand users or fans.

Generally, for a specific thing or event, once the classification criterion is appropriate, the consumer's opinions or behaviours of the same group or category are generally the same or similar, whereas that of different groups or categories may be significantly different. Additionally, all the different groups of consumers are living in the same world, they are interacting with each other and their opinions or behaviours are changing due to the influence of the members from all groups, which is just similar to the population dynamics among different species in biological systems.

Since population dynamics is a basic characteristic of consumers, many phenomena related to consumers also reflect some kind of such characteristic, thus models of biological population dynamics can be utilised to explain them well. For example, the famous Bass (1969) model, which described technology diffusion, was origin from the infection model. Thus, in the following, we try to use another famous population dynamics model, i.e., the Lotka-Volterra model, to explain the advertising's effect on sales, and prove that it fits the Little's five criteria well.

3.2 The advertising model based on Lotka-Volterra model

The system considered includes two competing firms which sell similar products and advertise for them. Assume their sales along time t are $x_1(t)$ and $x_2(t)$, and their advertising levels are $q_1(t)$ and $q_2(t)$ respectively. In this paper, we use a two-level framework to describe the advertising effect. Firstly, the population dynamics effects among consumers are described by a general Lotka-Volterra model as follows:

$$\begin{cases} \dot{x}_1 = x_1\left(b_1 - a_{11}x_1 - a_{12}x_2\right) \\ \dot{x}_2 = x_2\left(b_2 - a_{21}x_1 - a_{22}x_2\right) \end{cases} \tag{1}$$

Where b_1 and b_2 are the intrinsic sales growth rate of firm 1 and 2; a_{11} (or a_{22}) is restriction coefficient of firm 1's (or 2's) own products; a_{12} (or a_{21}) is competition coefficient of firm 2 (or 1) related to firm 1 (or 2). In detail, a_{12} refers to the impact on the firm 1's sales due to firm 2's sales. Analogously, a_{21} refers to the impact on the firm 2's sales due to firm 1's sales. a_{11}, a_{12}, a_{21}, a_{22}, are all positive constants. In addition, considering that the sales of its own usually have more effects on the limitation of its own sales than the sales of rivals, we assume that $a_{12} < a_{11}$, $a_{21} < a_{22}$, are hold. The item $a_{11}x_1$ (or $a_{22}x_2$) represents the sales restriction as the result from its own sales, whereas the item $a_{12}x_2$ (or $a_{21}x_1$) is the sales restriction in the result from the rival sales. The item b_1a_{21} (or b_2a_{12}) represents the competitive force related to advertising level of the firm 1 (or 2) to the firm 2 (or 1) respectively.

The intrinsic sales growth rate is generally affected by many factors such as price, quality, promotion and advertising. In this paper, we just focus on the impact of advertising and ignore other factors, we assume that

$$b_i = f\left(q_i\right), \quad i = 1, 2 \tag{2}$$

where $q_i = q_i(t)$ is the advertising level of firm i along time t. Generally, the function f is required to satisfy the following properties:

1 $f(0) > 0$, which implies even the firm does not advertise its product, it can also have a positive intrinsic sales growth rate due to other factors such as fair quantities and novel outlook

2 $\partial f / \partial q > 0$ which means that advertising has a positive effect on the intrinsic sales growth rate.

In fact, equations (1) and (2) form a new framework for analysing the effect of advertising, in which equation (1) illustrates the dynamics process of advertising's effects on consumer population, whereas equation (2) communicates the positive influence of advertising on the intrinsic sales growth rate.

For the new framework consisting of equation (1) and (2), let $q_i(t) = \overline{q}_i$ or $b_i(t) = f(\overline{q}_i) = \overline{b}_i$ fixed, which means the two firms keep their advertising level fixed as constants all the time, then the system will finally reach a stable equilibrium when $t \to +\infty$. When the two firms reach such a stable equilibrium (x_1^*, x_2^*), the market shares or sales of the two firms will not change, i.e., $dx_i(t) / dt = 0$, and thus we have

$$\begin{cases} x_1^*\left(b_1 - a_{11}x_1^* - a_{12}x_2^*\right) = 0 \\ x_2^*\left(b_2 - a_{21}x_1^* - a_{22}x_2^*\right) = 0 \end{cases} \tag{3}$$

Solving equation (3), we get four possible equilibrium points: $O(0, 0)$, $P(b_1 / a_{11}, 0)$, $Q(0, b_2 / a_{22})$, and $M((a_{22}b_1 - a_{12}b_2)/(a_{11}a_{22} - a_{12}a_{21}), (a_{11}b_2 - a_{21}b_1)/(a_{11}a_{22} - a_{12}a_{21}))$.

4 Model analysis

A good advertising response model is necessary to be utilised to explain many advertising effects. Summarising the advertising effects which were proved by empirical studies, Little (1979) proposed five basic criteria which aggregate advertising models should satisfy. In this section, we will test whether our model fits these five criteria to verify the applicability of our model.

The five criteria proposed by Little (1979) are as follows.

P1 sales respond dynamically upward and downward to increases and decreases of advertising and frequently do so at different rates

P2 steady-state response can be concave or S-shaped and will often have positive sales at zero advertising

P3 competitive advertising affects sales

P4 the dollar effectiveness of advertising can change over time as the result of changes in media, copy, and other factors

P5 products sometimes respond to increased advertising with a sales increase that falls off even as advertising is held constant.

To check whether our model fits the five criteria mentioned above, we analyse our model in the following two different conditions, i.e.,

1 the response to rectangular pulse of advertising

2 the steady state responding to steady advertising.

4.1 Response to rectangular pulse of advertising

Assume that the two companies' sales have already reached the steady state at time $t = 0$ conditioning that the advertising levels of the two firms have been kept as constants q_{10} and q_{20} for a long time respectively, i.e.,

$$\begin{cases} \dot{x}_1\big|_{t=0} = x_{10}\left(b_{10} - a_{11}x_{10} - a_{12}x_{20}\right) = 0 \\ \dot{x}_2\big|_{t=0} = x_{20}\left(b_{20} - a_{21}x_{10} - a_{22}x_{20}\right) = 0 \end{cases} \tag{4}$$

in which x_{10} and x_{20} are their initial sales at time $t = 0$, $b_{10} = \mathrm{f}(q_{10})$, $b_{20} = f(q_{20})$ are the intrinsic sales growth rate of the two firms.

When firm 1 increases its advertising level to $q_{10} + \delta$ where $\delta > 0$, we have $b_1 = f(q_{10} + \delta) > b_{10}$. Denote $b_1 = b_{10} + \Delta$, then

$$\dot{x}_1 = x_{10}\left(b_{10} + \Delta - a_{11}x_{10} - a_{12}x_{20}\right) \tag{5}$$

in which $\Delta > 0$ reflects the increment of the intrinsic sales growth rate of firm 1 due to its advertising. Substituting the first formula in equation (4) into equation (5) and calculating, we can get

$$\dot{x}_1 = x_{10}\Delta > 0 \tag{6}$$

which means as long as firm 1 improves its advertising level, the steady state will be broken and the sales of firm 1 will increase.

Assume that firm 1 keeps this new advertising level $q_{10} + \delta$ at the interval $0 < t < T$ and then reduces to its initial level q_{10}, we obtain the limit on the left and the limit on the right of the changing rate of firm 1's sales at the point of transition T as follows:

$$\begin{cases} \dot{x}_1\big|_{t=T^-} = x_1(T)\left(b_{10} + \Delta - a_{11}x_1(T) - a_{12}x_2(T)\right) > 0 \\ \dot{x}_1\big|_{t=T^+} = x_1\left(b_{10} - a_{11}x_1(T) - a_{12}x_2(T)\right) < 0 \end{cases} \tag{7}$$

Observing equation (7), it is clear that the sales respond dynamically upward and downward to increases and decreases of advertising. Further, when

$$0 < T < \left(2b_{10} + \Delta - 2a_{11}x_{10} - 2a_{12}x_{20}\right) / 2a_{11}\dot{x}_{10} \tag{8}$$

we can get

$$\dot{x}_1\big|_{t=T^-} + \dot{x}_1\big|_{t=T^+} = 2x_1(T)\left(b_{10} - a_{11}x_1(T) - a_{12}x_2(T)\right) + x_1(T)\Delta > 0 \tag{9}$$

which means that when the firm advertises its product within a certain time, the rate of increase is higher than the rate of decrease all along. The results meet the criterion P1, which is shown as the curve of firm 1 in Figure 1.

Figure 1 Sales response to rectangular pulse of advertising

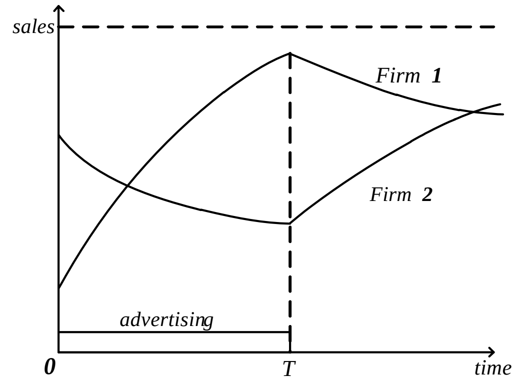


Figure 1 also illustrates the following facts.

1 during the period that firm 1 increases its advertising level at $0 < t < T$, the market share of firm 2 changes with the changing of its rival advertising level and thus our model can fit the criterion P3

2 during the advertising period $0 < t < T$, the effectiveness of advertising to firm 1 decreases along time t

Thus, if we look upon q_i as some other factors which affect the intrinsic sales growth rate b_i such as carrying new media and copy, our model can display the criterion P4.

4.2 Steady state response with steady advertising

When both the two firms keep their advertising levels fixed as constants, the system will finally reach a stable equilibrium. As discussed in Section 2, four possible steady states may finally be achieved. Considering the conditions of the four states (Ahmad, 1993), we get

$$x_1(\infty) = \begin{cases} 0 & \text{if} \quad 0 < b_1 \le a_{12}b_2/a_{22} \\ (a_{22}b_1 - a_{12}b_2)/(a_{11}a_{22} - a_{12}a_{21}) & \text{if} \quad a_{12}b_2/a_{22} < b_1 < a_{11}b_2/a_{21} \\ b_1/a_{11} & \text{if} \quad b_1 \ge a_{11}b_2/a_{21} \end{cases} \tag{10}$$

The relationship between the final stable sales and the intrinsic sales growth rate of firm 1 which is given by equation (10) is shown as Figure 2.

Figure 2 shows the following facts:

1 when a firm's sale growth rate is really low (i.e., $b_1 < a_{12}b_2 / a_{22}$), its stable equilibrium sale is zero, which implies the firm will eventually be pushed out of the market by its competitor

2 once the firm's sale growth rate is larger than a certain level (i.e., $b_1 < a_{12}b_2 / a_{22}$), its sale will increase sharply

3 when the sale growth of a firm is extremely large (i.e., $b_1 < a_{11}b_2 / a_{21}$), it will push its competitor out of the market.

Noting that the intrinsic sales growth rate will be affected by the advertising ($b_i = f(q_i)$), we change the x-axis from the intrinsic sales growth rate to the advertising level and then obtain Figure 3.

Figure 2 Steady state sales response to intrinsic sales growth rate

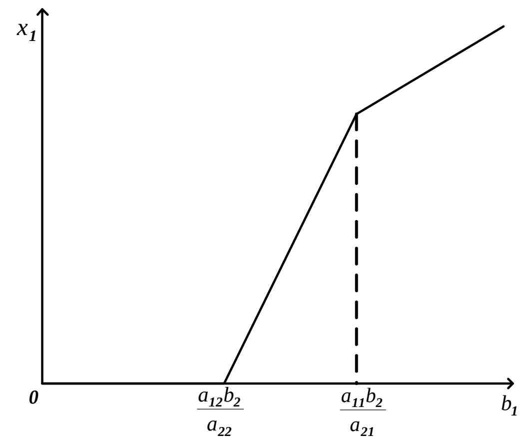
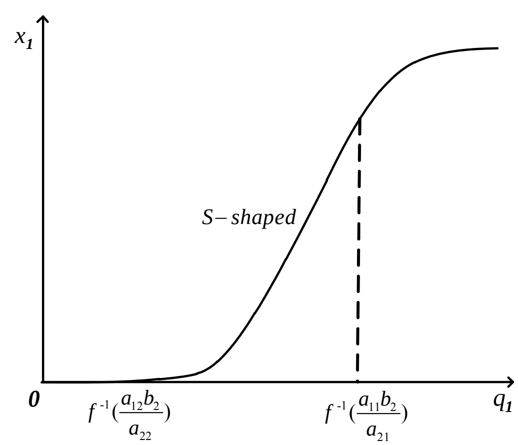


Figure 3 Steady state sales response to advertising level



In drawing Figure 3, we utilise a special form of $f(q_i)$ as:

$$f(q_i) = \left(A_1 + A_2 e^{\theta q_i}\right) / \left(1 + e^{\theta q_i}\right), \quad i = 1, 2, \tag{11}$$

in which $A_2 > A_1 > 0$, $\theta > 0$ are all constants. For the $f(q_i)$ given by equation (11), we have $f'(q) = \theta(A_2 - A_1)e^{\theta qi} / (1 + e^{\theta qi})^2 > 0$, which means the advertising level has a positive impact on the intrinsic sales growth rate. Furthermore, we can prove that there exists an upper bound for the $f(q_i)$ given by equation (11), i.e., $f(q_i) < A_2$. It illustrates the fact that advertisements cannot improve the intrinsic sales growth rate infinitely.

Figure 3 illustrates the following facts.

1 the advertising's effect on sales is S-Shaped, which the latter is just the second criterion P2 of Little (1979)

2 the effect of the advertising may exist a threshold effect, i.e., when a firm's advertising level is rather low (i.e., $q_1 \le f^{-1}(a_{12}b_2/a_{22})$), its stable sales are zero, whereas the sales will increase shapely when $q_1 > f^{-1}(a_{12}b_2/a_{22})$

3 noting that the advertising's effect has an upper bound, i.e., $f(q_i) < A_2$, we get that the advertising will never improve the firm's sales if $A_2 < a_{12}b_2/a_{22}$.

In this case, the advertising is useless and the entire advertising budget seems as being input into a black hole. On the contrary, if $f(0) > a_{12}b_2 / a_{22}$, the firms can always get a part of market share even it invests nothing in advertising. The above two facts imply that other factors such as product quality, price are also important for a firm's survival.

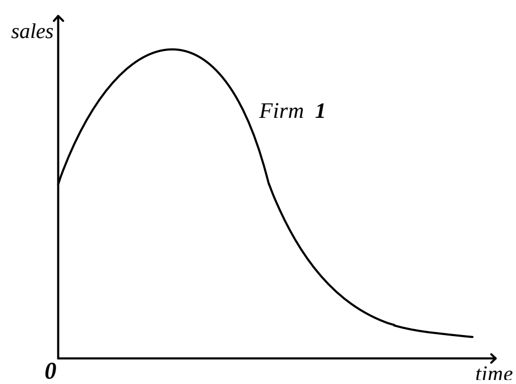
4.3 *The Lotka-Volterra model with finite delays*

The above analysis shows that the proposed model can satisfy the former four criteria, i.e., P1 to P4. When we test whether the proposed model which consists of equation (1) and (2) fits the last criterion P5, it fails. However, this problem can be solved if we take into account the facts that the rival's effects are usually delayed. To achieve the object, we modify the standard Lotka-Volterra model given by equation (1) into a Lotka-Volterra model with finite delays, i.e.,

$$\begin{cases} \dot{x}_1 = x_1\left(b_1 - a_{11}x_1 - a_{12}\int_{(t-\tau,0)^+}^{t} \omega(t-s)x_2(s)\mathrm{d}s\right) \\ \dot{x}_2 = x_2\left(b_2 - a_{21}\int_{(t-\tau,0)^+}^{t} \omega(t-s)x_1(s)\mathrm{d}s - a_{22}x_2\right) \end{cases} \tag{12}$$

In equation (12), $\tau > 0$ is the finite delay time, $\omega(\cdot) > 0$ is a decreasing weight function which implies a large effect occurs for the most recent event. For the modified model, when we keep the advertising level fixed in the period $0 < t < T$, we get the first firm's sales as shown in Figure 4. Figure 4 shows that the modified model satisfies the last criterion P5.

Figure 4 The sales of the modified model response to fixed advertising



Finally, we summarise our analysis results for the model proposed in this paper and the Little's analysis results for other classic models such as Vidale-Wolfe model, Nerlove-Arrow model and Lanchester model in Table 1. In Table 1, the symbol '√' means the model fits the criterion related well, the symbol '√–' implies the model can flexibly satisfy the criterion if we change the original model in some way, and the symbol '×' indicates the model does not fit that criterion. From this summary, it is easy to find that our model has superiorities in explaining the advertising effects than others.

Table 1 Comparison with former advertising response models

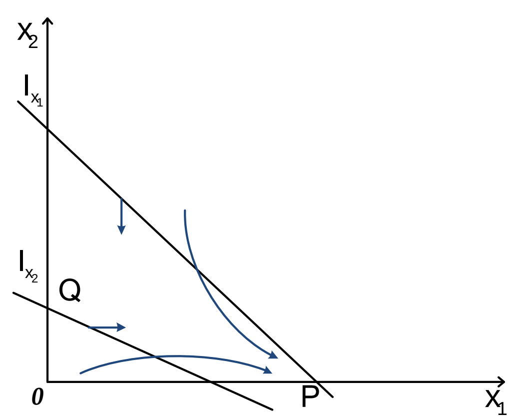
	P1	*P2*	*P3*	*P4*	*P5*
Vidale-Wolfe	√	×	×	×	×
Nerlove-Arrow	×	√–	√–	√–	×
Lanchester	√	√–	√	√	×
Lotka-Volterra	√	√	√	√	√–

5 Stable equilibrium analysis and marketing crisis warning

As illustrated in Section 2, four possible stable equilibriums exist for the proposed model, $O(0, 0)$, $P(b_1/a_{11}, 0)$, $Q(0, b_2/a_{22})$ and $M((a_{22}b_1 - a_{12}b_2)/(a_{11}a_{22} - a_{12}a_{21}), (a_{11}b_2 - a_{21}b_1)/(a_{11}a_{22} - a_{12}a_{21}))$.

The first equilibrium point $O(0, 0)$ is a stable equilibrium if and only if $b_1 < 0$ and $b_2 < 0$, which implies that both the intrinsic sales growth rate of the two firms can never be positive. Due to the evolution of technology, some products may be out of time and be eventually vanished in the market. Under such conditions that $b_1 < 0$ and $b_2 < 0$, advertising will be useless. Noting that we have assumed that $f(q_i)$ is always positive, this equilibrium has been driven out from our model.

Figure 5 Track trend of the second equilibrium point (see online version for colours)

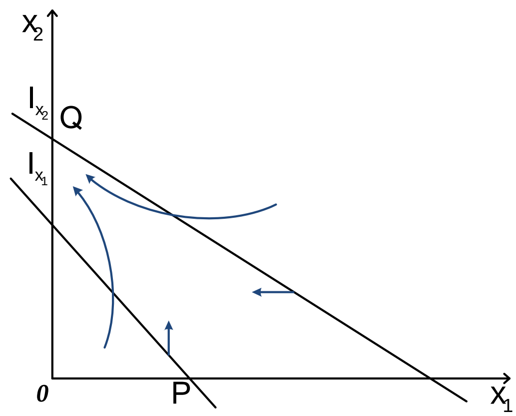


The second equilibrium point P is the intersection point of the straight line $x_2 = 0$ and $l_{x_1} = b_1 - a_{11}x_1 - a_{21}x_2$ (see Figure 5). This equilibrium will exist if and only if

$b_1 > a_{11}b_2 \,/\, a_{21}$, which implies the firm 1's intrinsic sales growth rate b_1 is extremely larger than that of its competitor b_2. Under such condition the firm 1 will finally occupy the whole market $b_1 \,/\, a_{11}$.

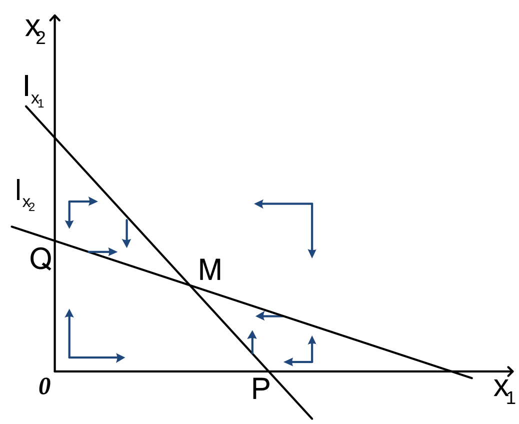
The third equilibrium point Q the intersection point of the straight line $x_1 = 0$ and $l_{x_2} = b_2 - a_{21}x_1 - a_{22}x_2$ (see Figure 6), which is just contrary to that of the second scenario. This equilibrium will be reached when $0 < b_1 \le a_{12}b_2 \,/\, a_{22}$ holds. Under such condition, the firm 1 will finally be pushed out of the market whereas the firm 2 will monopolise the market with a sale of $b_2 \,/\, a_{22}$.

Figure 6 Track trend of the third equilibrium point (see online version for colours)



The last equilibrium point M is the intersection point of l_{x_1} and l_{x_2} indicating that the two firms coexist in the market and the sales of the two firms are $(a_{22}b_1 - a_{12}b_2)/(a_{11}a_{22} - a_{12}a_{21})$ and $(a_{11}b_2 - a_{21}b_1)/(a_{11}a_{22} - a_{12}a_{21})$ respectively (see Figure 7). This stable equilibrium will exist when $(a_{12}b_2 \,/\, a_{22} < b_1 < a_{11}b_2 \,/\, a_{21})$, which means both the two forms have moderate intrinsic sales growth rate.

Figure 7 Track trend of the fourth equilibrium point (see online version for colours)



Noting that the equilibrium sale of a firm is positive linear with its intrinsic sales growth rate while the intrinsic sales growth rate of a firm increases by its advertising level, we know that the advertising has positive impacts on the sales. Further, we can prove that $(a_{22}b_1 - a_{12}b_2)/(a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_2 - a_{21}b_1)/(a_{11}a_{22} - a_{12}a_{21}) > b_1 / a_{11}$ and $(a_{22}b_1 - a_{12}b_2)/(a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_2 - a_{21}b_1)/(a_{11}a_{22} - a_{12}a_{21}) > b_2 / a_{22}$, which means the whole market capacity in the duopoly market is larger than the market capacity of one firm in the oligopoly market.

As argued by Laitinen (2009), management would benefit in decisions from simple rules of thumb more than from complicated models, to consider the firm's possible final stable status will be more important than to know its exact evolution process reaching that result. Since the firm 1 may eventually be pushed out of the market once $0 < b_1 \leq a_{12}b_2 / a_{22}$ or $0 < q_1 \leq f^{-1}(a_{12}b_2 / a_{22})$ is held if the two firms keep their advertising level fixed permanently, the value of $f^{-1}(a_{12}b_2 / a_{22})$ can be looked on as a warning value of the firm 1's marketing crisis. Once the firm 1 finds that its advertising level is lower than $f^{-1}(a_{12}b_2 / a_{22})$, it should evaluate whether it can break through the adverse condition by investing more in advertisement. If the answer is yes, the firm should calculate the minimum and the optimal input on advertising. On the contrary, the firm should consider other strategies such as improving the product quality, reducing the price or even investing in R&D to a new product.

For example, suppose that the $b_i = f(q_i)$ satisfies equation (11), and other parameters have been obtained by some ways as follows: $a_{11} = 0.11$, $a_{12} = 0.1$, $a_{21} = 0.1$, $a_{22} = 0.12$, $A_1 = 2$, $A_2 = 4$, $\theta = 0.02$. Assuming that the firm 1 has measured that the advertising level of its competitor was $q_2 = 90$, whereas its own level is $q_1 = 9$ during the past years. From the above parameters, the firm can obtain the following results: $b_1 = 3.09$, $b_2 = 3.716$, $a_{12}b_2 / a_{22} = 3.097$. Note that $b_1 < a_{12}b_2 / a_{22}$, the firm 1 knows that it may be pushed out of the market by its competitor someday if nothing is changed in the current situation for a while.

Therefore, the firm 1 has to improve its advertising level, and the minimum advertising level for its survival in the market is $q_{1\_low} = f^{-1}(a_{12}b_2 / a_{22})$= 9.7, and the advertising level for it getting half of the market is $q_{1\_half} = 61.5$.

6 Conclusions

Based on the consumer's population dynamics, this paper proposes a new advertising response model framework. In the proposed framework, the general Lotka-Volterra model is utilised to describe population dynamics among consumers, whereas a general increasing function is used to reflect the advertisement's effects on the intrinsic sales growth rate of a product. Analysis of the new model shows that the proposed model can fit all the five criteria summarised by Little (1979), and thus has more advantages than other classic advertising response models such as Vidale-Wolfe model, Nerlove-Arrow model, Lanchester model and their modifications. Also, the stable equilibrium of the proposed model has been analysed and an early-warning marketing mechanism is proposed based on these equilibriums.

Contribution of this paper including the following:

1 we propose a new aggregate advertising model to describe the effects of advertising on sales, which can make us understand the advertising's effects from the angle of consumer population dynamics

2 we prove that the proposed model satisfies all the five criteria summarised by Little (1979), which implies our model can explain most advertising phenomena

3 we introduce an early-warning marketing mechanism based on the equilibrium analysis of our model, which can make the firm react to the marketing strategy of its competitors quickly.

Under the new framework of the proposed model, future works of this study can be extended from the following several ways:

1 to utilise the empirical method such as panel data regression (Luo and Wang, 2008) to verify the model and to determine the values of the parameters of our model such as a_{11}, a_{12}, a_{21}, a_{22}

2 to apply the model into practise and use the results to depict the evolution of firms' sales or market shares

3 to generalise the model from two firms to multiple competing firms

4 to apply our models to the firm's advertising and pricing decision in a supply chain framework (Xiao et al., 2010; Xiao and Yan, 2011).

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grand No. 70901068), the Funds for International Cooperation and Exchange of the National Natural Science Foundation of China (Grant No. 71110107024), Anhui Provincial Natural Science Foundation (No. 090416240) and Chinese Universities Scientific Fund. Liang Liang and Qinglong Gou would also like to acknowledge the Science Fund for Creative Research Groups of the National Natural Science Foundation of China (Grant No. 71121061) for support of their research.

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