Closed-Loop Advertising Strategies in a Duopoly

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Using the Lanchester model to describe the dynamics of the market where two firms compete for customers by advertising, we solve the problem of determining an optimal advertising strategy for maximum discounted profits. We develop both open- and closed-loop strategies and explain the relationship between them. Using a new mathematical approach, we prove that our closed-loop solution is a global Nash equilibrium. The closed-loop strategy is time-variant and depends linearly on the actual market share. The time-variant coefficient incorporates the discount factor; its computation requires the solution of a backward differential equation and a set of two nonlinear differential equations for an initial value problem. The closed-loop advertising expenditures are proportional to the open-loop advertising expenditures and to the square of the competitor's actual market share. This provides a very practical adaptive control rule that allows the manager to adjust the actual advertising expenditure and to deviate from budget. We illustrate the use of our control rule, using data for the period 1968–1984 of the Cola War. Marketing implications of the results are provided.
(*Marketing—Competitive Strategy; Nash Equilibrium; Bilinear-Quadratic Differential Game; Noncooperative*)

1. Introduction

In our increasingly competitive and dynamic world, with a growing emphasis on marketing, companies are spending ever more on advertising. Although many recent studies suggest that companies are spending too much on advertising (Eastlack and Rao 1989, Ibrahim and Lodish 1993), executives are often quoted as saying that they have no choice, because that is what the competition is doing.

Our objective in this paper is to exercise parsimony in modeling the process, so that an analytical solution is obtained, but not at the expense of richness in approximating the main phenomena in the market. The main features that we want to capture are the dynamics of the market on the one hand, and competitive and strategic behavior on the other.

Like many previous studies, we have chosen the Lanchester model to describe the market dynamics. Though simple, it is rich enough to provide a meaningful description of aggregate market sales. Indeed, it has been used by many researchers, both in marketing and other social fields, to model a wide variety of competitive situations (Kimball 1957, Vidale and Wolfe 1957, Isaacs 1965, Horsky 1977, Little 1979, Case 1979, Deal et al. 1979, Deal 1979, Erickson 1985).

Although the model is rich in describing the effect of marketing activities on sales, in order to investigate what will actually happen in a given market, one has to model the objectives and strategies of the competing firms, and find the dynamic equilibrium that will evolve in the market. The straightforward approach is to assume that each firm maximizes its own discounted expected cash flow (net present value—NPV). This has indeed been the approach taken by several recent publications in the field. Chintagunta and Vilcassim (1992), and Erickson (1992), for example, have applied this model, together with the solution outlined in Case

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(1979), to solve the case of the Cola War between Coke and Pepsi. Both have found that the closed-loop solution provides a better fit to the data than an open-loop solution. However, their results are restricted to the particular case of a zero discount rate.

In this paper we reconsider the problem stated by Chintagunta and Vilcassim (1992), and Erickson (1992), of finding competitive advertising strategies which maximize the NPV in a duopoly market with dynamics described by the Lanchester model. We provide a solution for both the open-loop and the closed-loop advertising strategy for Nash equilibrium. Our solution generalizes the previously published results in several directions. First, in contrast to previous literature related to this problem (Case 1979, Erickson 1992, Chintagunta and Vilcassim 1992), we are able to treat the case of nonzero discount rate.[1] We do this by considering a wider strategy set, i.e., time-variant closed-loop strategies that may depend on initial conditions. Using a new optimization approach—completing the objective function to a perfect square—we prove that our closed-loop strategy is a global Nash equilibrium, for the general case of nonzero discount. Our approach avoids the mathematical computations associated with the Hamilton-Jacobi-Bellman equations. We also find necessary and sufficient conditions for which the time-variant closed-loop strategy will coincide with the open-loop strategy and become time-invariant. Finally, the closed-loop solution has a very nice property. The advertising expenditure of the closed-loop is proportional to the open-loop advertising expenditure (by which expenditure is adjusted over time) and to the square of the competitor's actual market share. Thus it is an easily implementable control rule.

In the following, we first describe the model and solve it mathematically, then relate it to previous work, and show how it works with the parameters generated by the Cola War.

2. Problem Statement

Assume the dynamic system given by the "combat equation," known as the Lanchester model,

$$\dot{x}(t) = \rho_1[1 - x(t)]u_1(t) - \rho_2 x(t)u_2(t), \quad x(0) = x_0. \tag{1}$$

Assuming a fixed pool of customers, then the terms $x(t)$ and $1 - x(t)$, represent the fractions of the total customer pool that, at time t, purchase the firm's and the competitor's product, respectively. The control variables $u_1(t)$ and $u_2(t)$ represent the square root of the firm's and the competitor's marketing expenditures, respectively. The effectiveness of the promotion efforts, in the simultaneous combat on customers, is measured by $\rho_1 u_1(t)$ for the firm and by $\rho_2 u_2(t)$ for the competitor. The constants ρ_1 and ρ_2 are related to media-buying, and to other product and market characteristics.

Consider now the following standard discounted profit objective functions for the competitors

$$\prod_k (u_1, u_2) = \int_0^\infty [q_k x_k(t) - r_k u_k^2(t)]e^{-\mu t}dt, \quad k = 1, 2, \tag{2}$$

where

$$x_k = \begin{cases} x, & k = 1, \\ 1 - x, & k = 2. \end{cases} \tag{3}$$

The constant q_k, $k = 1, 2$, represents the gross profit rate of firm k, μ is the constant discount rate; and r_k, $k = 1, 2$ represents the effectiveness of advertising buying-power (perhaps, the two firms can get different discount rates on advertising). Usually, $r_1 = r_2 = 1$.

The admissible control u_k, $k = 1, 2$ is restricted to be a state feedback, (i.e., a function of x), bounded by the total budget of the company. More exactly, we consider a wider strategy set, where the control depends on (x_0, x, t) and find the closed-loop optimal control (see, e.g., Fershtman 1987, p. 219). In the special case where the initial state coincides with the current state of the system our solution becomes a feedback strategy which depends on (x, t).

The objective of this paper is to consider the time-variant bilinear-quadratic game problem of finding admissible u_k^*, $k = 1, 2$, satisfying the following inequality conditions:

$$\prod_1 (u_1^*, u_2^*) \geq \prod_1 (u_1, u_2^*) \quad \text{and} \tag{4}$$

$$\prod_2 (u_1^*, u_2^*) \geq \prod_2 (u_1^*, u_2), \tag{5}$$

[1] Sorger (1989) solved the nonzero discount rate case for a different dynamical model.

for *all* admissible u_k, $k = 1, 2$. In other words, we want to find an admissible strategy (u_1^*, u_2^*) that is a *global Nash equilibrium* of the differential game associated with (1) and (2).

3. Determination of the Closed-Loop Nash Equilibrium Advertising Strategies

To succeed in a dynamic market, advertising strategy must reflect the current situation. It is good practice to divide the planning cycle into two phases. In the analysis and planning phase, a marketing plan is elaborated. In the executive phase, the plan is adapted to the situation as monitored by the marketing manager.

In this work, we develop a simple approach that, when applied to an open-loop strategy, readily generates a closed-loop strategy. This allows a time-variant (open-loop) strategy (whose formulation may require elaborate calculations) to be developed in advance for the entire planning period; then, during the implementation phase, when time is at a premium, and information on real events becomes available, strategy can be updated.

The structure of this section is as follows. First we find an open-loop solution from the first-order necessary conditions. Then we construct the closed-loop solution, and prove the Nash equilibrium conditions (4) and (5). Finally, we show the simple transformation formula for the closed-loop strategy from the open-loop strategy.

3.1. The Algorithm of Constructing the Closed-Loop Strategies

In the following, we develop first-order necessary conditions for optimality for a set of control variables which are explicit functions of time, i.e., $u_1(t)$ and $u_2(t)$. In the next stage, we construct the closed-loop strategies and then prove that they satisfy the Nash equilibrium inequalities (4) and (5).

Considering the differential game (1) and (2) and using a variational approach, see Appendix 1 (A3, A4, A6), cf. Bryson and Ho (1975), we obtain the following first-order necessary conditions:

$$\dot{\lambda}_k(t) = \mu\lambda_k(t) + (\rho_1 u_1(t) + \rho_2 u_2(t))\lambda_k(t) + (-1)^k q_k, \lim_{t\to\infty} \lambda_k(t)e^{-\mu t} = 0, \quad k = 1, 2, \quad (6)$$

$$u_k(t) = \frac{(-1)^{k+1}}{2} r_k^{-1}\rho_k\lambda_k(t)(1 - x_k(t)), \quad k = 1, 2, \quad (7)$$

where $\lambda_k(t)e^{-\mu t}$ represents the Lagrange multiplier.

Substituting (7) in (1) and (6), the following two-point boundary-value problem (TPBVP) is obtained:

$$\dot{x} = \tfrac{1}{2}[r_1^{-1}\rho_1^2(1 - x)^2\lambda_1 + r_2^{-1}\rho_2^2 x^2\lambda_2], \quad x(0) = x_0, \quad (8)$$

$$\dot{\lambda}_k = \mu\lambda_k + \tfrac{1}{2}[r_1^{-1}\rho_1^2\lambda_1\lambda_k(1 - x) - r_2^{-1}\rho_2^2\lambda_2\lambda_k x] - (-1)^{k+1}q_k, \lim_{t\to\infty} \lambda_k(t)e^{-\mu t} = 0, \quad k = 1, 2. \quad (9)$$

To find the values for u_k that produce a stationary value for Π_k, we must solve the TPBVP (8) and (9). Note that Deal (1979), as well as Chintagunta and Vilcassim (1992), treated this TPBVP for the case $\mu = 0$. They were able to find only a numerical solution. In the following, we will give an analytical solution.

Notation. In the following exposition we will use the notation x^P for x which solves (8) and (9). The superscript "P" stands for "Plan."

LEMMA 1. *Let λ_k be as in* (8) *and* (9) *and suppose*

$$\lambda_k(t) = (-1)^{k+1}q_k e^{\mu t}\Phi(t), \quad k = 1, 2. \quad (10)$$

Then with the above notation, Φ satisfies the following backward differential equation:

$$\Phi'(t) = \tfrac{1}{2}[r_1^{-1}\rho_1^2 q_1(1 - x^P) + r_2^{-1}\rho_2^2 q_2 x^P]e^{\mu t}\Phi^2(t) - e^{-\mu t}, \quad \Phi(\infty) = 0. \quad (11)$$

REMARK 1. From Lemma 1, it follows that (10) is a solution of (9) and the set of two equations in (9) can be reduced to Equation (11).

PROOF OF LEMMA 1. Equation (11) follows immediately by substituting (10) and its derivative with respect to t into (9). □

COROLLARY 1. *Let $\Phi(t)$ be as in* (11) *and suppose*

$$\Phi(t)e^{\mu t} = \psi(x^P). \quad (12)$$

Then ψ satisfies the following differential equation[2]:

[2] If $\mu = 0$, using the transformation $\psi^2(x^P) = f(x^P)$, Equation (13) becomes a linear differential equation; integrating it and considering (10) and (12), we obtain

$$\lambda_k(t) = (-1)^{k+1}q_k\psi(x^P) = (-1)^{k+1}q_k \frac{[4(x^P(\infty) - x^P)]^{1/2}}{[r_1^{-1}\rho_1^2 q_1(1 - x^P)^2 - r_2^{-1}\rho_2^2 q_2(x^P)^2]^{1/2}} \quad k = 1, 2.$$

For finding $x^P(\infty)$, one solves the equation $\dot{x}^P = 0$.

$$\psi'(x^P)\psi(x^P)[r_1^{-1}\rho_1^2q_1(1-x^P)^2 - r_2^{-1}\rho_2^2q_2(x^P)^2] - [r_1^{-1}\rho_1^2q_1(1-x^P) + r_2^{-1}\rho_2^2q_2x^P]\psi^2(x^P) = 2\mu\psi(x^P) - 2, \quad \lim_{t\to\infty}\psi(x^P(t))e^{-\mu t} = 0. \tag{13}$$

PROOF. Equation (13) follows from (11) and the relation

$$\Phi'(t)e^{\mu t} + \mu\psi(x^P) = \psi'(x^P)\dot{x}^P. \quad \square \tag{14}$$

COROLLARY 2. *The TPBVP (8) and (9) can be transformed into the following initial value problem (IVP)*[3]:

$$\dot{x}^P = \tfrac{1}{2}[r_1^{-1}\rho_1^2(1-x^P)^2q_1 - r_2^{-1}\rho_2^2(x^P)^2q_2]e^{\mu t}\Phi(t), \quad x(0) = x_0, \tag{15a}$$

$$\Phi'(t) = \tfrac{1}{2}[r_1^{-1}\rho_1^2q_1(1-x^P) + r_2^{-1}\rho_2^2q_2x^P]e^{\mu t}\Phi^2(t) - e^{-\mu t}, \quad \Phi(0) = \psi(x_0), \tag{15b}$$

where $\psi(x_0)$ is obtained by solving the backward Equation (13).

PROOF. Equation (15a) follows from (8) by considering (10). Integrating (13) we obtain the value of $\psi(x_0)$. Considering (12) we obtain $\psi(x_0) = \Phi(0)$. Therefore the backward differential equation (11) can be transformed into the forward differential Equation (15b). $\square$

ASSUMPTION 1. *Let*

$$u_k^{OL} = \tfrac{1}{2}r_k^{-1}\rho_kq_k\Phi(t)e^{\mu t}(1 - x_k^P), \quad k = 1, 2, \tag{16}$$

be the open-loop strategies, where

$$x_k^P = \begin{cases} x^P, & \text{if } k = 1, \\ 1 - x^P, & \text{if } k = 2, \end{cases}$$

and, x^P and $\Phi(t)$ satisfy (15a–b).

ASSUMPTION 2. *Let*

$$u_k^* = \tfrac{1}{2}r_k^{-1}\rho_kq_k\Phi(t)e^{\mu t}(1 - x_k), \quad k = 1, 2, \tag{17}$$

be the closed-loop strategies, where x_k is the state as in (3) *and satisfies Equation* (1) *and, x^P and $\Phi(t)$ satisfy* (15a–b).

[3] An IVP is much easier to solve then a TPBVP.

From (17), we learn that the closed-loop strategies are time-variant and depend linearly on the actual market share of the competition, and nonlinearly on the market and brand parameters. The time-variant coefficient is computed at x^P, through Equations (15a–b) and (13).

REMARK 2. The time-variant coefficient, $e^{\mu t}\Phi(t)$, of both closed-loop (open-loop) advertising strategies, is index-invariant. This means that both firms' strategies react in the same way to time factors. At each moment, these strategies differ only by their effectiveness, gross profit rate and actual market share.

In the next subsection, we prove that the closed-loop strategies u_k^*, $k = 1, 2$, defined in (17), are global Nash equilibrium strategies, i.e. they satisfy inequalities (4) and (5).

3.2. Global Nash Equilibrium Closed-Loop Strategies: The Main Result

THEOREM 1. *Consider the differential game associated with* (1) *and* (2). *Then the pair (u_1^*, u_2^*) defined in* (17) *forms a global Nash equilibrium closed-loop strategy of the above differential game, i.e., it satisfies the conditions* (4) *and* (5) *for every admissible u_k, $k = 1, 2$.*

PROOF. See Appendix 2.

Note that we have found a closed-loop equilibrium which is Nash and is a function of (t, x_0, x). Since it does not necessarily constitute an equilibrium for a game that starts at a different x_0 it is a Nash equilibrium which is not subgame perfect.

REMARK 3. The input (17) is a closed-loop strategy with a very special structure. It uses the same "time-shape" as in the open-loop strategy and updates the changing in market conditions by exchanging $x^P(t)$ with the actual market share $x(t)$ as measured at real time t (and assuming here that it can be modeled by the dynamic equation (1)). The subsection 3.5. is devoted to further discussion on the relationship between the open-loop strategy u_k^{OL} and the optimal closed-loop strategy u_k^*.

3.3. The Trajectory of x^P: Properties

Let

$$a(x^P) = r_1^{-1}\rho_1^2(1-x^P)^2q_1 - r_2^{-1}\rho_2^2(x^P)^2q_2 = 0. \tag{18}$$

Considering (18) and (15a) we have

$$\dot{x}^P = a(x^P)\Phi(t)e^{\mu t}, \quad x^P(0) = x_0. \tag{19}$$

From (19) it follows that $x^P(t)$ increases or decreases with $\mathrm{sgn}[a(x^P)]$. Let us assume for simplicity that $r_1^{-1}\rho_1^2 = r_2^{-1}\rho_2^2$. Then considering (18) we conclude with the following:

(i) $x^P(t)$ increases or decreases with $\mathrm{sgn}[x^P(\infty) - x_0]$.

(ii) $\mathrm{sgn}[x^P(\infty) - 0.5] = \mathrm{sgn}(q_1 - q_2)$ and if $q_1 = q_2$, then $x^P(\infty) = 0.5$.

In other words, for firms with symmetric advertising effectiveness, $x^P(t)$ increases if the initial state is below the steady state and vice versa. Also if $q_1 > q_2$, then the steady state is greater than 0.5 and vice versa. For $q_1 = q_2$, the steady state is exactly 0.5.

3.4. The Open- and Closed-Loop Strategies: Computation

Equations (16) and (17) can be derived by integrating (13) and (15a–b).[4]

3.5. The Open Loop vs. the Closed Loop: Discussion

Comparison of (16) with (17) reveals similarities between closed-loop and open-loop advertising strategies. In fact, we have the following interesting relationship between them:

$$u_k^* = u_k^{OL}\left(\frac{1 - x_k}{1 - x_k^P}\right), \quad k = 1, 2. \tag{20}$$

The relationship (20) stands an important practical control rule; if one derives the open-loop solution, as explained in §§3.3 and 3.4, then the closed-loop solution is immediately obtained by multiplying the open-loop solution with the ratio between the market share of the competition, as measured at real time and the amount $(1 - x^P)$. For example, if Firm 1 discovers a 40% increase in the competitor market share with respect to the competitor planned market share $(1 - x^P)$, the advertising expenditure will be approximately doubled (1.4^2).

Note that at $t = 0$, the closed-loop strategies are equal to the open-loop strategies. This means that the closed-loop strategy uses the open-loop strategy at the initial time and then it works according the actual measurement x_k.

In the particular case, when $x(t) = x^P(t)$, the closed-loop strategy continues to work as the open-loop strategy. As can be seen from (20), this is a *necessary and sufficient condition* for the equality of the open- and closed-loop strategies. In this case, the closed-loop solution becomes time-invariant.

Summarizing this section we conclude with the following important results:

- The time-variant closed-loop strategies in (17) form a global Nash equilibrium for the differential game associated with (1) and (2). It depends linearly on the actual market share. The time-variant coefficient incorporates the discount factor.
- The time-variant coefficient, $e^{\mu t}\Phi(t)$, is the same for both players and requires the solution of a backward differential equation and a set of two nonlinear differential equations for an initial value problem.
- The closed-loop strategy is related to the open-loop strategy by the simple formula in (20).
- Necessary and sufficient conditions are given for which the closed-loop strategy becomes equal to the open-loop strategy. In this case the closed-loop solution becomes time-invariant.

[4] If $\mu = 0$, considering (16), (17), (12) and footnote 2, the following explicit form can be obtained for the open- and closed-loop strategies,

$$u_k^{OL} = \frac{1}{2} r_k^{-1}\rho_k q_k \psi(x^P)(1 - x_k^P)$$
$$= \frac{1}{2} r_k^{-1}\rho_k q_k \frac{[4(x^P(\infty) - x^P)]^{1/2}}{[r_1^{-1}\rho_1^2 q_1(1 - x^P)^2 - r_2^{-1}\rho_2^2 q_2(x^P)^2]^{1/2}}(1 - x_k^P),$$
$$k = 1, 2$$

and

$$u_k^* = \frac{1}{2} r_k^{-1}\rho_k q_k \psi(x^P)(1 - x_k)$$
$$= \frac{1}{2} r_k^{-1}\rho_k q_k \frac{[4(x^P(\infty) - x^P)]^{1/2}}{[r_1^{-1}\rho_1^2 q_1(1 - x^P)^2 - r_2^{-1}\rho_2^2 q_2(x^P)^2]^{1/2}}(1 - x_k),$$
$$k = 1, 2.$$

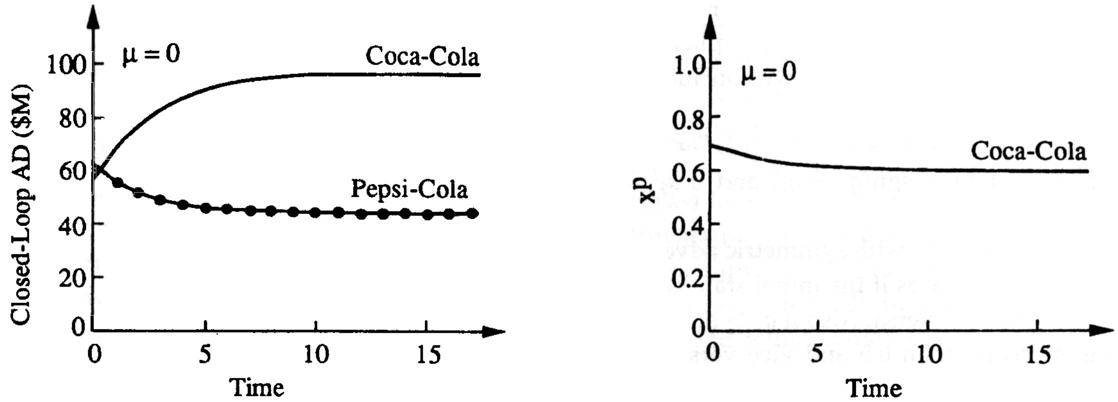
Also, considering (19), we have

$$\dot{x}^P = \tfrac{1}{2}[4(x^P(\infty) - x^P)a(x^P)]^{1/2}, \quad x^P(0) = x_0.$$

Integrating this equation, we obtain that x^P is a function of x_0 and t, and satisfies:

$$\int^{x^P} [4(x^P(\infty) - x)a(x)]^{-1/2}dx = \tfrac{1}{2}t.$$

Figure 1 Open-Loop Solution for the Parameters of the Cola War Left: Advertising Trajectory over Time Right: The Trajectory of x^P



4. Illustrative Examples, Comparisons with the Literature and Concluding Comments

In the following presentation, we solve and plot our results for the Cola War, where we use the data and parameters reported in Erickson (1992). The illustrations in Figure 1 are for the open-loop strategies and the trajectory of x^P. Using the relationship (20), between the open- and closed-loop strategy, we find the corresponding closed-loop strategy; and demonstrate (Figure 3) how our time-variant solution, incorporating the nonzero discount factor provides a better approximation to what happened in practice (Figure 2) than do the available solution in the literature.

Figure 1 solves the base case, i.e., $r_1 = r_2 = 1$, $\rho_1 = \rho_2 = 0.0119$, $x_0 = 0.694$, $q_1 = 795$, $q_2 = 366$, and $\mu = 0$. In this case $x^P(\infty) = 0.595766$. It shows, on the left, the advertising expenditures strategies over time, and, on the right, the corresponding x^P. Both strategies and x^P exhibit monotonic behavior. Coke, starting with a market share that is higher than the steady state and having a higher gross profit rate, increases its advertising over time as its market share slips to the steady-state level. Pepsi, on the other hand, starts with an advertising blitz, reducing it over time as it approaches steady state. At steady state, as we expect, the ratio of advertising expenditures is proportional to the ratio of the corresponding gross profits and the companies' market shares

Figure 2 Actual Advertising for the Cola War Case and Coke's Market Share

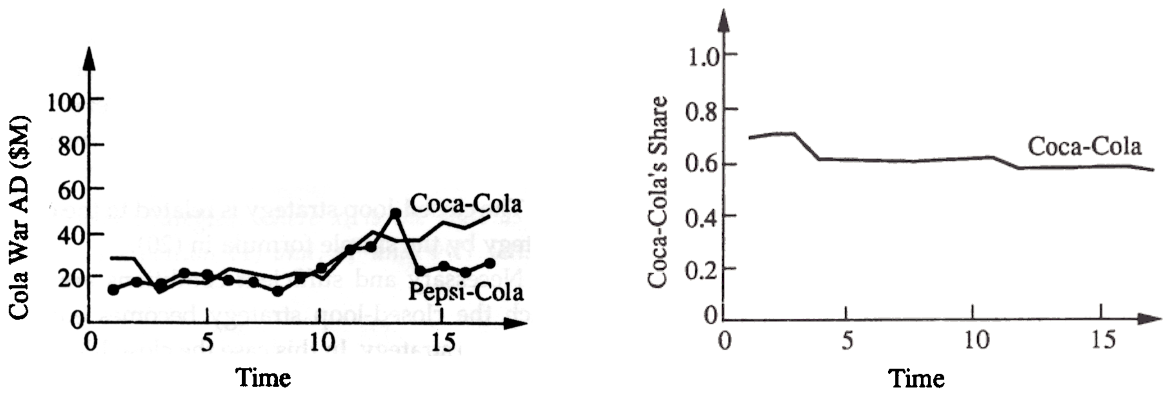
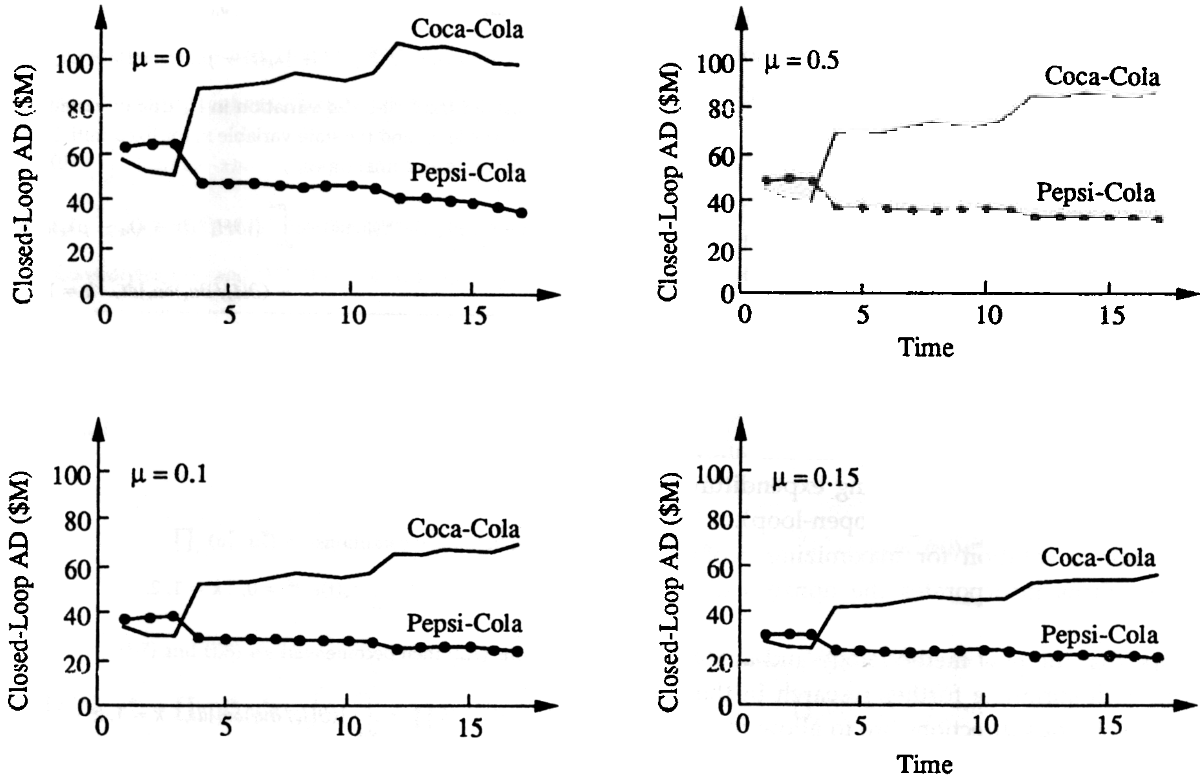


Figure 3 Closed-Loop Advertising for Cola War for Different Discount Rates



(Coke's expenditure is approximately twice the size of Pepsi's expenditure).

Figure 2 shows the actual data for the Cola War during the period 1968–1984 (the year before the introduction of the "New Coke"). The left part shows the advertising expenditures for the two brands, and the right part, Coke's market share.

Figure 3 shows our closed-loop advertising expenditures for different discount rates, based on the above actual market share. We see that advertising levels are lower for both firms when discount rate is higher. This make sense since firms invest less in advertising that generates future return. Furthermore, we see that the closed-loop solution does indeed seem to fit Coke's actual advertising better than does the open-loop. This finding matches the results of Erickson (1992), which evaluated the case of zero discount rate. Our new results show that the approximation is better for a nonzero discount rate. In fact, $\mu = 15\%$ provides the closest fit. This would indeed be the representative discount rate over this whole period, taking into account inflation plus risk-adjusted return. (We realize that the discount rate may have changed over the years, e.g. due to different inflation rates, but a thorough econometrics analysis of this point is beyond the scope of this paper.)

In summary, this section illustrates the following points:

1. Open-loop advertising is monotonic. It starts with an amount proportional to the company's gross profit rate and the initial market share of the competitor and it decreases as market share increases, and vice versa. It converges to an amount proportional to the company's gross profit rate and to the steady-state market share of the competitor. The steady-state market share depends on the gross profit rate, where the higher the relative gross profit, the higher is the steady-state market share.

2. The higher the discount rate, the lower is the advertising spending.

3. The closed-loop strategy is no longer monotonic, since it relates to what is happening to the actual market

share. We show that the closed loop with a 15% discount rate provides a better approximation to what happened in the Cola War than the previously solved case of zero discount.

5. Conclusions

This paper presents a new approach to analyzing dynamic competitive problems. It provides for the development of optimal closed-loop strategies that avoid the mathematical computations associated with the Hamilton-Jacobi-Bellman equations. In addition the following contribution are obtained:

- The special construction of closed-loop strategies provides a practical adaptive control rule that allows the manager to adjust actual advertising expenditure and deviate from the planned budget (open-loop).
- The optimal solution for maximizing profits for competitive firms incorporates the nonzero discount rate.

We hope that the new methodology and analysis of this paper will stimulate further research in the area. Some of the obvious directions are to allow for disturbances in the system and in the measurements, for uncertainty in the parameters, and to estimate the parameters using a filtering and adaptive control approach.[5]

[5] The authors wish to thank Professor Gary Erickson for providing us with the empirical data used in Figure 2, and for valuable comments on previous drafts, and the reviewers and the Associate Editor for their constructive comments. The authors wish to thank Professors Arkadi Nemirovski and Dan Peled for valuable discussions.

Appendix 1

Derivations of the First-Order Necessary Conditions for Nash Equilibrium

Adjoining the constraint Equation (1) to Π_k, $k = 1, 2$, with a Lagrange multiplier, $\lambda_k(t)e^{-\mu t}$, $k = 1, 2$, we obtain,

$$\overline{\Pi}_k = \int_0^\infty \{q_k x_k(t) - r_k u_k^2(t) + \lambda_k(t)[\rho_1(1 - x(t))u_1(t) - \rho_2 x(t)u_2(t) - \dot{x}(t)]\}e^{-\mu t}dt, \quad k = 1, 2. \tag{A1}$$

For convenience, we define a scalar function H_k, $k = 1, 2$ (the Hamiltonian), as follows:

$$H_k(x(t), u_1(t), u_2(t), \lambda_k(t)) = \{q_k x_k(t) - r_k u_k^2(t) + \lambda_k(t)[\rho_1(1 - x(t))u_1(t) - \rho_2 x(t)u_2(t)]\}e^{-\mu t}.$$

Then, integrating the term on the right side of (A1) by parts, yields

$$\overline{\Pi}_k = -(\lim_{t\to\infty} \lambda_k(t)e^{-\mu t})x(\infty) + \lambda_k(0)x_0 + \int_0^\infty \{H_k(x(t), u_1(t), u_2(t), \lambda_k(t)) + [\dot{\lambda}_k(t) - \mu\lambda_k(t)]x(t)e^{-\mu t}\}dt, \quad k = 1, 2.$$

Now consider the first-order variation in $\overline{\Pi}_k$ due to variations in the control variable u_k, and the state variable x, for fixed initial conditions and fixed initial and final times,

$$\delta\,\overline{\Pi}_k = -(\lim_{t\to\infty} \lambda_k(t)e^{-\mu t})\delta x(\infty) + \int_0^\infty \{[\partial H_k/\partial x + (\dot{\lambda}_k - \mu\lambda_k)e^{-\mu t}]\delta x + (\partial H_k/\partial u_k)\delta u_k\}dt, \quad k = 1, 2. \tag{A2}$$

Define λ_k, so as $k = 1, 2$, to cause the coefficients of δx to vanish, that is, as

$$\dot{\lambda}_k = \mu\lambda_k - \frac{\partial H_k}{\partial x}e^{\mu t} = \mu\lambda_k + (\rho_1 u_1 + \rho_2 u_2)\lambda_k + (-1)^k q_k, \quad k = 1, 2, \tag{A3}$$

with the boundary conditions

$$\lim_{t\to\infty} \lambda_k(t)e^{-\mu t} = 0, \quad k = 1, 2. \tag{A4}$$

Equations (A2) then become

$$\delta\,\overline{\Pi}_k = \int_0^\infty [(\partial H_k/\partial u_k)\delta u_k]dt, \quad k = 1, 2. \tag{A5}$$

For the extreme, $\delta\overline{\Pi}_k$, $k = 1, 2$, must be zero for an arbitrary δu_k, $k = 1, 2$; this can only happen if

$$\frac{\partial H_k}{\partial u_k} = [-2r_k u_k - (-1)^k \rho_k(1 - x_k)\lambda_k]e^{-\mu t} = 0, \quad k = 1, 2,$$

or

$$u_k = \frac{(-1)^{k+1}}{2} r_k^{-1}\rho_k\lambda_k(1 - x_k), \quad k = 1, 2. \tag{A6}$$

Equations (A3), (A4) and (A6) are the Euler-Lagrange equations.

Appendix 2

Proof of Theorem 1

The idea is to transform the payoff functional $\Pi_k(u_1, u_2)$, $k = 1, 2$, to a perfect square, by adding a suitable zero sum.

Consider the zero sum

$$0 = -q_k x_k \Phi(t)|_0^\infty + \int_0^\infty \frac{d[q_k x_k \Phi(t)]}{dt}dt, \tag{B1}$$

where $\phi(t)$ is as in (15b) or (11), and x_k as in (1)–(3).

Considering (B1) we obtain

$$0 = q_k x_0^k \Phi(0) + \int_0^\infty [q_k \dot{x}_k \Phi(t) + q_k x_k \dot{\Phi}(t)]dt. \tag{B2}$$

Using Equation (1) we obtain

$$0 = q_k x_0^k \Phi(0) + \int_0^\infty \{[\rho_1(1-x)u_1 - \rho_2 x u_2](-1)^{k+1} q_k \Phi(t) + q_k x_k \dot{\Phi}(t)\}dt. \quad \text{(B3)}$$

Now, consider Equation (15b) or (11), then (B3) will become

$$0 = q_k x_0^k \Phi(0) + \int_0^\infty \{[\rho_1(1-x)u_1 - \rho_2 x u_2](-1)^{k+1} q_k \Phi(t) + \tfrac{1}{2} q_k x_k [r_1^{-1}\rho_1^2 q_1(1-x^P) + r_2^{-1}\rho_2^2 q_2 x^P] e^{\mu t}\Phi(t)^2 - q_k x_k e^{-\mu t}\}dt. \quad \text{(B4)}$$

Let $k = 1$. Considering (16) and (17), then (B4) will become

$$0 = q_1 x_0 \Phi(0) + \int_0^\infty [2r_1 u_1^* u_1 + 2r_2(q_1/q_2)u_2^*(u_2^{OL} - u_2) - 2r_1 u_1^{OL} u_1^* + \rho_1 q_1 u_1^{OL}\Phi(t)e^{\mu t} - q_1 x]e^{-\mu t}dt. \quad \text{(B5)}$$

Now let $k = 2$, then considering again (B4), (16) and (17), we obtain

$$0 = q_2(1-x_0)\Phi(0) + \int_0^\infty \{2r_2 u_2^* u_2 + 2r_1(q_2/q_1)u_1^*[(\rho_2/\rho_1)u_2^{OL} - u_1] - 2r_2(\rho_1/\rho_2)u_1^{OL}u_2^* + \rho_1 q_2 u_1^{OL}\Phi(t)e^{\mu t} - q_2(1-x)\}e^{-\mu t}dt. \quad \text{(B6)}$$

Now adding the zero sum (B5) to Equation (2), we obtain

$$\begin{aligned}\prod\nolimits_1 (u_1, u_2) &= q_1 x_0 \Phi(0) + \int_0^\infty -r_1 u_1^2 + 2r_1 u_1^* u_1 + 2r_2(q_1/q_2)u_2^*(u_2^{OL} - u_2) - 2r_1 u_1^{OL}u_1^* + \rho_1 q_1 u_1^{OL}\Phi(t)e^{\mu t}]e^{-\mu t}dt \\ &= q_1 x_0 \Phi(0) + \int_0^\infty -r_1(u_1^* - u_1)^2 + r_1 u_1^{*2} + 2r_2(q_1/q_2)u_2^*(u_2^{OL} - u_2) - 2r_1 u_1^{OL}u_1^* + \rho_1 q_1 u_1^{OL}\Phi(t)e^{\mu t}]e^{-\mu t}dt. \end{aligned} \quad \text{(B7)}$$

Particularly,

$$\prod\nolimits_1 (u_1^*, u_2^*) = q_1 x_0 \Phi(0) + \int_0^\infty [r_1 u_1^{*2} + 2r_2(q_1/q_2)u_2^*(u_2^{OL} - u_2^*) - 2r_1 u_1^{OL}u_1^* + \rho_1 q_1 u_1^{OL}\Phi(t)e^{\mu t}]e^{-\mu t}dt. \quad \text{(B8)}$$

Now considering (B7) and (B8) we have

$$\begin{aligned}\prod\nolimits_1 (u_1, u_2^*) &= \prod\nolimits_1 (u_1^*, u_2^*) - \int_0^\infty r_1(u_1^* - u_1)^2 e^{-\mu t}dt \\ &\le \prod\nolimits_1 (u^*1, u_2^*),\end{aligned} \quad \text{(B9)}$$

for every admissible u_1, which is exactly condition (4).

Now adding the zero sum (B6) to Equation (2), we obtain

$$\begin{aligned}\prod\nolimits_2 (u_1, u_2) &= q_2(1-x_0)\Phi(0) \\ &\quad + \int_0^\infty \{-r_2 u_2^2 + 2r_2 u_2^* u_2 + 2r_1(q_2/q_1)u_1^*[(\rho_2/\rho_1)u_2^{OL} - u_1] - 2r_2(\rho_1/\rho_2)u_1^{OL}u_2^* + \rho_1 q_2 u_1^{OL}\Phi(t)e^{\mu t}\}e^{-\mu t}dt \\ &= q_2(1-x_0)\Phi(0) \\ &\quad + \int_0^\infty \{-r_2(u_2^* - u_2)^2 + r_2 u_2^{*2} + 2r_1(q_2/q_1)u_1^*[(\rho_2/\rho_1)u_2^{OL} - u_1] - 2r_2(\rho_1/\rho_2)u_1^{OL}u_2^* + \rho_1 q_2 u_1^{OL}\Phi(t)e^{\mu t}\}e^{-\mu t}dt. \end{aligned} \quad \text{(B10)}$$

Particularly,

$$\begin{aligned}\prod\nolimits_2 (u_1^*, u_2^*) &= q_2(1-x_0)\Phi(0) \\ &\quad + \int_0^\infty \{r_2 u_2^{*2} + 2r_1(q_2/q_1)u_1^*[(\rho_2/\rho_1)u_2^{OL} - u_1^*] \\ &\quad 2r_2(\rho_1/\rho_2)u_1^{OL}u_2^* + \rho_1 q_2 u_1^{OL}\Phi(t)e^{\mu t}\}e^{-\mu t}dt.\end{aligned} \quad \text{(B11)}$$

Now considering (B10) and (B11) we have

$$\prod\nolimits_2 (u_1^*, u_2) = \prod\nolimits_2 (u_1^*, u_2^*) - \int_0^\infty r_2(u_2^* - u_2)^2 e^{-\mu t} \le \prod\nolimits_2 (u_1^*, u_2^*), \quad \text{(B12)}$$

for every admissible u_2, which is exactly condition (5). □

References

Bryson, A. E. and Y. C. Ho, *Applied Optimal Control*, Hemisphere Publishing Corporation, New York, 1975.

Case, J. H., Economics and the Competitive Process, New York University Press, New York, 1979.

Chintagunta, P. K. and N. J. Vilcassim, "An Empirical Investigation of Advertising Strategies in a Dynamic Duopoly," *Management Sci.*, 38 (1992), 1230–1244.

Deal, K., S. P. Sethi, and G. L. Thompson, "A Bilinear-quadratic Differential Game in Advertising," in Liu and Satinen (Eds.), *Control Theory in Mathematical Economics*, Marcel Dekker, New York, 1979, 91–109.

——, "Optimizing Advertising Expenditures in a Dynamic Duopoly," *Oper. Res.*, 27, 4 (1979), 682–692.

Eastlack, J., Jr. and A. Rao, "Advertising Expenditures at the Campbell Soup Company," *Marketing Sci.*, 8, 1 (1989).

Erickson, G. M., "A Model of Advertising Competition," *J. Marketing Res.*, 22 (1985), 297–304.

——, "Empirical Analysis of Closed-loop Duopoly Advertising Strategies," *Management Sci.*, 38 (1992), 1732–1749.

Fershtman, C., "Identification of Classes of Differential Games for Which the Open Loop is a Degenerate Feedback Nash Equilibrium," *J. Optimization Theory and Application*, 55, 2 (1987), 217–231.

Horsky, D., "An Empirical Analysis of the Optimal Advertising Policy," *Management Sci.*, 23 (1977), 1037–1049.

Ibrahim, M. and L. Lodish, "How Advertising Works," I.R.I. Research Paper, City, 1993.

Isaacs, R., "Differential Games," John Wiley, New York, 1965.

Kimball, G. E., "Some Industrial Applications of Military Operations Research Methods," *Oper. Res.*, 5 (1957), 201–204.

Little, J. D. C., "Aggregate Advertising Models: The State of the Art," *Oper. Res.*, 27 (1979), 629–667.

Sorger, G., "Competitive Dynamic Advertising: A Modification of the Case Game," *J. Economic Dynamics and Control*, 13 (1989), 55–80.

Vidale, M. L. and H. B. Wolfe, "An Operations Research Study of Sales Response to Advertising," *Oper. Res.*, 5 (1957), 370–381.

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