Competitive Advertising under Uncertainty:

A Stochastic Differential Game Approach

Ashutosh Prasad and Suresh P. Sethi*

The University of Texas at Dallas, Richardson, Texas 75083-0688

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* Corresponding author. Tel.: 972-883-6245. Fax.: 972-883-2799. Email: sethi@utdallas.edu. The authors thank Janos Turi and Ph.D. seminar participants at UT Dallas for helpful comments.

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Abstract:

We analyze optimal advertising spending in a duopolistic market where each firm's market share depends on its own and its competitor's advertising decisions, and is also subject to stochastic disturbances. We develop a differential game model of advertising in which the dynamic behavior is based on the classic Vidale-Wolfe advertising model and the Lanchester model of combat, as well as perturbed by a Brownian motion. Particularly important to note is the morphing of the Vidale-Wolfe sales decay term into decay caused by competitive advertising and non-competitive 'churn' that acts to equalize market shares in the absence of advertising. We derive closed-loop Nash equilibria for symmetric as well as asymmetric competitors. For all cases, explicit solutions and comparative statics are presented.

Keywords: Advertising expenditure; Advertising budgeting; Competitive strategy; Differential games; Stochastic calculus

1. Introduction

The advertising spending decision has been the focus of considerable interest for researchers in marketing as evidenced by the large body of literature devoted to this subject starting with Vidale and Wolfe (1957) and reported in surveys by Sethi (1977) and Feichtinger, Hartl and Sethi (1994). The annual expenditure on advertising by firms is very large, with a CMR/TNS (2003) media report that total ad spending for all media was 117 billion dollars in 2002 in the US alone. At the same time, marketers have noted that firms often advertise in a suboptimal manner. For example, Patti and Blasko (1981) and Blasko and Patti (1984) found in surveys that a large percentage of industrial and consumer goods firms do advertising budgeting based on the affordable method, the percentage-of-sales method, and the competitive-parity method. These methods are rarely optimal. A number of researchers have also concluded that firms tend to over-advertise (Aaker and Carmen 1982, Lodish et al. 1995).

The combination of the large amounts of money spent on advertising and potential inefficiencies in the advertising budgeting process motivates the interest in better understanding optimal advertising budgeting. However, one must take care to limit the conclusions of optimality to only those markets for which the model applies. For instance, ad expenditure or advertising policies that are optimal in a monopoly setting would not be optimal in a competitive setting. We, thus, define our market context and research question as follows.

We examine a duopoly market in a mature product category where the two firms compete for market share using advertising as the dominant marketing tool. The firms are strategic in their behavior, that is, they take actions that maximize their objective while also considering the actions of the competitor. Furthermore, they interact dynamically for the foreseeable future. This is in part due to the carry-over effect of advertising, which means that advertising spending today will continue to influence sales several months down the line. Each firm's advertising acts to increase its market share while the competitor's advertising acts to reduce its market share. In addition to competitive effects, market share decay, or churn, can also be caused by non-competitive effects described in the next paragraph. However, marketing and competitive activities alone do not govern market shares in a deterministic manner because there is inherent randomness in the marketplace and in the choice behavior of customers. The market for cola drinks, dominated by Coke, Pepsi and their Cola Wars, provides us with an example of a market with such features (Chintagunta and Vilcassim 1992, Erickson 1992, Fruchter and Kalish 1997).

We now consider non-competitive reasons for market share churn. (In a duopoly situation, the decay of market share for one firm is a gain in market share for the other. Thus, 'churn' rather than 'decay' is a more appropriate term and is used hereafter.) Although a firm's market share churn is a dynamic effect that occurs continuously, it is obscured during the period of advertising by market share gains made due to advertising. It is hence, most visible when the firm does not advertise. Explanations for non-competitive churn are product obsolescence, forgetting (Vidale-Wolfe 1957), lack of market differentiation (Bain 1956), lack of information (Nelson 1974), variety seeking (McAlister and Pessemier 1982) and brand switching. Absent stochasticity, if market share churn were due solely to competitive effects, consider what would happen if neither firm advertised – market shares would remain at a fixed level, never changing. However, due to the non-competitive factors just mentioned, one would expect market shares to converge to a long-run equilibrium when neither brand is advertised for a very long duration. The proposed model takes into account decay due to competitive advertising as well as churn due to non-competitive factors.

For a competitive market with stochastic disturbances and other features as described above, our objective then is to recommend optimal advertising expenditures over time for the two firms. Due to the carry-over effect of advertising, the optimal advertising spending over time is determined using dynamic optimization methods.[1] In the present case, a stochastic differential game model is formulated that is based on a monopoly model due to Sethi (1983), which is stochastic and explicitly solvable. We find a unique equilibrium where the optimal advertising for both firms follows a simple rule. We also find that market shares will be Beta distributed and that average market shares take the form of an attraction model. Finally, an illustration of the results is provided.

Our research follows in the operations research tradition in marketing. Similar advertising expenditure problems have been examined, for example, by Chintagunta (1993), Chintagunta and Vilcassim (1992), Deal, Sethi and Thompson (1979), Erickson (1995, 1991), Fruchter and Kalish (1997), Horsky (1977), Horsky and Mate (1988) and Sorger (1989). Whereas elements of the marketing environment described above, such as dynamics, competition, competitive and non-competitive decay, and also stochasticity, have been commonly accepted and described by individual models, there have been few

[1] The techniques used in this paper are discussed in textbooks on dynamic optimization, including Sethi and Thompson (2000). A useful reference for stochastic processes is Karlin and Taylor (1981). For a discussion of stochastic calculus in marketing, see Jain and Raman (1990).

attempts to study them together. We provide a focused literature discussion in the next section.

The rest of the paper is divided into sections dealing with the existing literature, the description of the model, the analysis for symmetric and asymmetric firms, and, finally, the conclusions.

2. Background

Among the earliest aggregate response models is the Vidale-Wolfe model whose dynamics are given by

$$\frac{dx(t)}{dt} = \rho u(t)(1 - x(t)) - \delta x(t), \quad x(0) = x_0, \tag{1}$$

where $x(t)$ is the sales rate (expressed as a fraction of the total market) at time t, $u(t)$ is the advertising expenditure rate, ρ is a response constant and δ is a market share decay constant. ρ determines the effectiveness of advertising while δ determines the rate at which consumers are lost due to product obsolescence, forgetting, etc. The formulation has several desirable properties, for example, market share has a concave response to advertising, and there is a saturation level (Little 1979). Sethi (1973) and others have provided the optimal advertising path for this type of problem.

Subsequent research has concentrated on extending the basic framework to include the effect of advertising expenditure by competitors. Dynamic advertising competition among duopolists battling for market share has been investigated by, among others, Deal (1979), Deal, Sethi and Thompson (1979), Erickson (1995, 1992) and Chintagunta and Vilcassim (1992), and surveyed by Erickson (2003) and Jorgensen

(1982). These studies have used the Lanchester model of combat to characterize the market share evolution over time for the competing firms (Kimball 1957, Little 1979). Although there is no consensus on the competitive extension of the Vidale-Wolfe formulation, a possible dynamics based on the Lanchester model of combat is

$$\begin{aligned}\frac{dx(t)}{dt} &= \rho_1 u_1(x(t), y(t))(1-x(t)) - \rho_2 u_2(x(t), y(t))x(t), \quad x(0) = x_0, \\ \frac{dy(t)}{dt} &= \rho_2 u_2(x(t), y(t))x(t) - \rho_1 u_1(x(t), y(t))(1-x(t)), \quad y(0) = 1 - x_0,\end{aligned} \tag{2}$$

where $x(t)$ and $y(t)$ represent the market shares of the two firms, whose parameters and decision variables are indexed 1 and 2 respectively.[2] Note that $x(t) + y(t) = 1$.

Differential games can be solved using either open-loop or closed-loop solution concepts (e.g., Chintagunta 1993, Erickson 1995, 1992, Feichtinger et. al 1989, Fruchter and Kalish 1997). In the open-loop solution, competing firms decide at inception what their advertising expenditures will be over the planning horizon. The closed-loop solution envisages that competing firms decide upon their advertising response given the current state. Whereas this solution concept is intuitively more appealing, robust, and satisfies what game theorists call subgame perfection, it is more difficult to compute a closed-loop solution versus an open-loop solution. Typically, resort must be made to numerical methods of solution. In this paper, however, we will obtain explicit closed-loop solutions.

In addition to competitive extensions, recent research has delved into the problem of stochastic disturbances where the state variable, usually market share, is determined by

[2] When advertising expenditure enters linearly in the dynamic equation, its cost in the objective function is often assumed to be quadratic (or more generally, convex) to ensure concavity of the objective function (e.g., Erickson 1995). Equivalently, one can take the square root of the advertising expenditure in the dynamic equation and subtract advertising expenditure linearly in the objective function (e.g., Sorger 1989). See Gould (1970) and Sethi and Thompson (2000) for a discussion. However, if empirical evidence suggests convexity of the objective function, see the literature on chattering or pulsing advertising policies (Mahajan and Muller 1986, Sasieni 1971)

stochastic disturbances in addition to advertising spending (e.g., Sethi 1983). We start with the stochastic, monopoly advertising formulation of Sethi (1983). This formulation is given by the Itô equation

$$dx(t) = \left(\rho u(t)\sqrt{1-x(t)} - \delta x(t)\right)dt + \sigma(x)dw(t), \quad x(0) = x_0, \tag{3}$$

where $\sigma(x)$ represents a variance term and $w(t)$ represents a standard Wiener process. The formulation has the useful feature in that it has a basic resemblance to the Vidale-Wolfe model and at the same time it permits an explicit solution to the advertising spending decision. We wish to extend this model to incorporate competition.

A related extension is due to Sorger (1989). He uses a special case of the Lanchester model to take advantage of Sethi's (1983) formulation that results in an explicit solution. This is,

$$\begin{aligned} \frac{dx}{dt} &= \rho_1 u_1(x,y)\sqrt{1-x} - \rho_2 u_2(x,y)\sqrt{x}, \quad x(0) = x_0, \\ \frac{dy}{dt} &= \rho_2 u_2(x,y)\sqrt{1-y} - \rho_1 u_1(x,y)\sqrt{y}, \quad y(0) = 1 - x_0. \end{aligned} \tag{4}$$

In particular, Sorger also describes the appealing characteristics of the model in detail, noting that it is compatible with word-of-mouth and nonlinear effects, and provides a comparison with other dynamics used in the advertising scheduling literature. However, the decay constant δ is not included in that model and it is assumed to be replaced totally by competitive effects.

On the other hand, we extend the Sethi model to allow for competition. We are able to do so while retaining the decay constant. Note that a stochastic version of Sorger's model is a special case of ours when $\delta = 0$. The decay constant which goes back to the Vidale-Wolfe formulation is not solely replaced by competitive advertising effects. Thus, in order to capture effects such as forgetting the decay parameter has been morphed into the churn parameter. We will discuss how including the term affects the outcome.

We will consider the case of symmetric and asymmetric competitors in a duopolistic market. The discussion focuses on the infinite horizon case. Although having a finite horizon presents no theoretical difficulties, it is unclear that additional insights would be forthcoming to balance the much greater complexity of the resulting mathematical expressions in the finite horizon case.

3. The model

We consider a duopoly market in a mature product category where total sales are distributed between the two firms, denoted firm 1 and firm 2, which compete for market share through advertising spending. We denote the market shares of firms 1 and 2 at time t as $x(t)$ and $y(t)$, respectively. Table 1 gives the additional notation with the subscript $i \in \{1, 2\}$ to reference the two firms.

<Insert Table 1 here>

The time argument will be suppressed in future where no confusion arises. The model dynamics are given by

$$\begin{aligned} dx &= [\rho_1 u_1(x,y)\sqrt{1-x} - \rho_2 u_2(x,y)\sqrt{x} - \delta(x-y)]dt + \sigma(x,y)dw, \quad x(0) = x_0, \\ dy &= [\rho_2 u_2(x,y)\sqrt{1-y} - \rho_1 u_1(x,y)\sqrt{y} - \delta(y-x)]dt - \sigma(x,y)dw, \quad y(0) = 1 - x_0. \end{aligned} \tag{5}$$

The specification of the dynamics given by equation (5) has the same desirable properties of concave response with saturation as the Vidale-Wolfe model. The market share is non-decreasing with own advertising, and non-increasing with the competitor's advertising expenditure. Consistent with literature, non-competitive decay is proportional to market share. As discussed, this churn is caused by influences other than competitive advertising, such as a lack of perceived differentiation between brands, so that market shares tend to converge in the absence of advertising. Finally, market shares are subject to a white noise $\sigma(x, y)dw$.

Since $dx + dy = 0$ and since $x(0) + y(0) = 1$, this implies that $x(t) + y(t) = 1$ for all $t \geq 0$. Now that $y(t) = 1 - x(t)$, we need only use the market share of firm 1 to completely describe the market dynamics. Thus, $u_i(x, y)$, $i = 1, 2$ and $\sigma(x, y)$ can be written as $u_i(x, 1-x)$ and $\sigma(x, 1-x)$. With a slight abuse of notation, we will use $u_i(x)$ and $\sigma(x)$ in place of $u_i(x, 1-x)$ and $\sigma(x, 1-x)$, respectively. Thus,

$$dx = [\rho_1 u_1(x)\sqrt{1-x} - \rho_2 u_2(x)\sqrt{x} - \delta(2x-1)]dt + \sigma(x)dw, \quad x(0) = x_0 \tag{6}$$

with $0 \leq x_0 \leq 1$.

As noted by Sethi (1983), an important consideration when choosing a formulation is that the market share should remain bounded within $[0,1]$ which can be problematic given stochastic disturbances. In our model it is easy to see that $x \in [0,1]$ almost surely (i.e., with probability 1) for $t > 0$, as long as $u_i(x)$ and $\sigma(x)$ are continuous functions which satisfy Lipschitz conditions on every closed subinterval of $(0,1)$ and further that

$$u_i(x) \geq 0, x \in [0,1] \tag{7}$$

and

$$\sigma(x) > 0, x \in (0,1) \text{ and } \sigma(0) = \sigma(1) = 0. \tag{8}$$

With (7) and (8), we have a strictly positive drift at $x = 0$ and a strictly negative drift at $x = 1$, i.e.,

$$\rho_1 u_1(0)\sqrt{1-0} + \delta > 0 \text{ and } -\rho_2 u_2(1) - \delta < 0. \tag{9}$$

Then from Gihman and Skorohod (1973) (Theorem 2, pp. 149, 157-158), $x = 0$ and $x = 1$ are natural boundaries for the solutions of (6) with $x_0 \in [0,1]$, i.e., $x \in (0,1)$ almost surely for $t > 0$.

Let m_i denote the industry sales volume multiplied by the per unit profit margin for firm i. The objective functions for the two firms are given by

$$\begin{aligned}
&\operatorname*{Max}_{u_1 \ge 0}\left\{V_1(x_0) = E\int_0^\infty e^{-r_1 t}[m_1 x(t) - c_1 u_1(t)^2]dt\right\},\\
&\operatorname*{Max}_{u_2 \ge 0}\left\{V_2(x_0) = E\int_0^\infty e^{-r_2 t}[m_2(1-x(t)) - c_2 u_2(t)^2]dt\right\},\\
&s.t.\\
&dx = [\rho_1 u_1(x)\sqrt{1-x} - \rho_2 u_2(x)\sqrt{x} - \delta(2x-1)]dt + \sigma(x)dw,\\
&x(0) = x_0 \in [0,1].
\end{aligned} \tag{10}$$

Thus, each firm seeks to maximize its expected, discounted profit stream subject to the market share dynamics.

4. Analysis

To find the closed-loop Nash Equilibrium strategies, we form the Hamilton-Jacobi-Bellman (HJB) equation for each firm:

$$r_1V_1 = \max_{u_1}\left\{ m_1x - c_1u_1^{\,2} + V_1'(\rho_1u_1\sqrt{1-x} - \rho_2u_2*\sqrt{x} - \delta(2x-1)) + \frac{\sigma(x)^2V_1''}{2} \right\}, \tag{11}$$

$$r_2V_2 = \max_{u_2}\left\{ m_2(1-x) - c_2u_2^{\,2} + V_2'(\rho_1u_1*\sqrt{1-x} - \rho_2u_2\sqrt{x} - \delta(2x-1)) + \frac{\sigma(x)^2V_2''}{2} \right\}, \tag{12}$$

where $V_i' = \frac{dV_i}{dx}, V_i'' = \frac{d^2V_i}{dx^2}$ and u_1* and u_2* denote the competitor's advertising policies in (11) and (12), respectively. We obtain the optimal feedback advertising decisions

$$u_1*(x) = \max\left(0, \frac{V_1'(x)\rho_1\sqrt{1-x}}{2c_1}\right) \text{ and } u_2*(x) = \max\left(0, -\frac{V_2'(x)\rho_2\sqrt{x}}{2c_2}\right). \tag{13}$$

Since $0 \le x \le 1$ and since it is reasonable to expect $V_1' \ge 0$ and $V_2' \le 0$, we can reduce the advertising decisions (13) to

$$u_1*(x) = \frac{V_1'(x)\rho_1\sqrt{1-x}}{2c_1} \text{ and } u_2*(x) = -\frac{V_2'(x)\rho_2\sqrt{x}}{2c_2}, \tag{14}$$

which hold as we shall see later. Substituting (14) in equations (11) and (12), we obtain the Hamilton-Jacobi equations

$$r_1V_1 = m_1x + \frac{V_1'^2\rho_1^{\,2}(1-x)}{4c_1} + \frac{V_1'V_2'\rho_2^{\,2}x}{2c_2} - V_1'\delta(2x-1) + \frac{\sigma(x)^2V_1''}{2}, \tag{15}$$

$$r_2V_2 = m_2(1-x) + \frac{V_2'^2\rho_2^{\,2}x}{4c_2} + \frac{V_1'V_2'\rho_1^{\,2}(1-x)}{2c_1} - V_2'\delta(2x-1) + \frac{\sigma(x)^2V_2''}{2}. \tag{16}$$

Following Sethi (1983), we attempt the following forms for the value functions

$$V_1 = \alpha_1 + \beta_1x \text{ and } V_2 = \alpha_2 + \beta_2(1-x). \tag{17}$$

These are inserted into equations (15) and (16) to determine the unknown coefficients $\alpha_1, \beta_1, \alpha_2, \beta_2$. Equating powers of x in equation (15) and powers of $1-x$ in equation

(16), the following four equations emerge, which can be solved for the unknown coefficients:

$$r_1\alpha_1 = \frac{\beta_1{}^2\rho_1{}^2}{4c_1} + \beta_1\delta, \tag{18}$$

$$r_1\beta_1 = m_1 - \frac{\beta_1{}^2\rho_1{}^2}{4c_1} - \frac{\beta_1\beta_2\rho_2{}^2}{2c_2} - 2\beta_1\delta, \tag{19}$$

$$r_2\alpha_2 = \frac{\beta_2{}^2\rho_2{}^2}{4c_2} + \beta_2\delta, \tag{20}$$

$$r_2\beta_2 = m_2 - \frac{\beta_2{}^2\rho_2{}^2}{4c_2} - \frac{\beta_1\beta_2\rho_1{}^2}{2c_1} - 2\beta_2\delta \tag{21}$$

A unique solution to these equations, together with the requirements that $\beta_1 > 0$ and $\beta_2 > 0$, will be shown to exist. Since for firms having different parameter values, the solutions are more complicated, we will first consider the case of two symmetric firms. The case of asymmetric firms will be dealt with in section 4.2.

4.1. Symmetric Firms

For this case, $\alpha = \alpha_1 = \alpha_2$, $\beta = \beta_1 = \beta_2$, $m = m_1 = m_2$, $c = c_1 = c_2$, $\rho = \rho_1 = \rho_2$ and $r = r_1 = r_2$. The four equations in (18-21) reduce to the following two:

$$\begin{aligned} r\alpha &= \frac{\beta^2\rho^2}{4c} + \beta\delta, \\ r\beta &= m - \frac{3\beta^2\rho^2}{4c} - 2\beta\delta. \end{aligned} \tag{22}$$

There are two solutions for β. One is negative, which clearly makes no sense. Thus, the remaining positive solution is the correct one. This also gives the corresponding α. The solution is

$$\alpha = \frac{(r-\delta)\left(W-\sqrt{W^2+12Rm}\right)+6Rm}{18Rr}, \beta = \frac{\sqrt{W^2+12Rm}-W}{6R}, \qquad (23)$$

where $R = \frac{\rho^2}{4c}, W = r + 2\delta$. We can now see that with the solution for the value function, the controls specified in equation (13) reduce to (14). This validates our choice of (14) in deriving the value function. Note that when the margin $m = 0$, the firm makes zero profit, i.e., the value functions $V_1 = \alpha + \beta x$ and $V_2 = \alpha + \beta(1-x)$ are identically zero. In turn, this implies that the coefficients α, β are each zero when $m = 0$.

Table 2 summarizes the analytical results and provides the comparative statics for the parameters on outcome variables, with the proofs in Appendix A. Although they are excluded from the table, $u_2{*}(y)$ and $V_2(y)$, $y = 1-x$, have the same comparative statics as $u_1{*}(x)$ and $V_1(x)$, respectively, due to symmetry.

<Insert Table 2 here>

When ρ increases or c decreases, i.e., there is a marginal increase in the value of advertising or a reduction in its cost, then, as one might expect, the amount of advertising increases. However, contrary to what one would expect to see in a monopoly model of advertising, the value function decreases. This occurs because in this market all advertising occurs from competitive motivations, since the optimal advertising expenditure would be zero if a single firm were to own both identical products. Advertising does not increase the size of the marketing pie but only affects its allocation. Thus, the increase in advertising causes a decrease in the value function.

The same logic does not apply when m increases, or r decreases. In these cases, it is true that the wasteful advertising is increased, but it is also true that the size of the pie has increased. Although intuitively it is difficult to predict that the latter effect should dominate the former, it turns out to be the case that an increase in m or decrease in r improves the value function.

The churn parameter δ reduces competitive intensity. Hence, it might be expected that an increase in δ should increase the profitability by reducing advertising. In fact, only the constant α part of the value functions increases and it is ambiguous what happens to the value functions overall. We can derive the exact conditions under which there is an increase or a decrease in the value function of a firm due to an increase in δ. We find that if the market share of a firm is less than half, the effect on the firm's value function is always positive. However, if the market share of a firm is greater than half, its value function can decrease because of an increase in δ if $x > \frac{\sqrt{(r+2\delta)^2+12Rm}-(r+2\delta)}{6r}+\frac{1}{2}$ is satisfied. The reason is that when a firm has a market share advantage over its rival, δ helps the rival unequally by tending to equalize market shares.

4.2. Asymmetric firms

We now return to the general case of asymmetric firms. For asymmetric firms, we re-express equations (18-21) in terms of a single variable β_1 which is determined by the solution to the quartic equation (24):

$$3R_1^{\ 2}\beta_1^{\ 4} + 2R_1(W_1 + W_2)\beta_1^{\ 3} + (4R_2m_2 - 2R_1m_1 - W_1^{\ 2} + 2W_1W_2)\beta_1^{\ 2} + 2m_1(W_1 - W_2)\beta_1 - m_1^{\ 2} = 0 \tag{24}$$

$$\alpha_1 = \frac{\beta_1}{r_1}(\beta_1R_1 + \delta) \tag{25}$$

$$\beta_2 = \frac{m_1 - \beta_1^{\ 2}R_1 - \beta_1W_1}{2\beta_1R_2} \tag{26}$$

$$\alpha_2 = \frac{\beta_2}{r_2}(\beta_2R_2 + \delta) \tag{27}$$

where $R_1 = \frac{\rho_1^{\ 2}}{4c_1}, R_2 = \frac{\rho_2^{\ 2}}{4c_2}, W_1 = r_1 + 2\delta, W_2 = r_2 + 2\delta$.

Once we obtain the correct value of β_1 out of the four solutions that will be obtained, the other coefficients can be obtained by solving for α_1 and β_2 and then, in turn, α_2. The solution is given in Appendix B.

We now collect the main results of the analysis into Proposition 1.

Proposition 1: For the advertising game described in (10):

(a) There exists a unique closed-loop Nash equilibrium solution to the differential game. (Proof in Appendix B)

(b) Optimal advertising is $u_1^*(x) = \frac{\beta_1\rho_1\sqrt{1-x}}{2c_1}$, $u_2^*(x) = \frac{\beta_2\rho_2\sqrt{1-y}}{2c_2}$*, where in the symmetric firm case, from equation (23),* $\beta_1 = \beta_2 = \frac{\sqrt{W^2 + 12Rm} - W}{6R}$*, and in the asymmetric firm case,* β_1 *and* β_2 *are given by (B14) and (26).*

We see that the optimal advertising policy is to spend in proportion to the competitor's market share. Consistent with Sorger (1989) and Erickson (1985), the firm that is in a disadvantageous position fights harder than its opponent and it should succeed in wresting market share from the opponent. Spending is decreasing in own market share, thus, the advertising-to-sales ratio is higher for the lower share firm. As noted in the introduction, many firms do advertising budgeting based on the affordable method, the percentage-of-sales method, and the competitive-parity method (Joseph and Richardson 2002, Patti and Blasko 1981, Blasko and Patti 1984). These methods would suggest that the firm with lower market share should spend less on advertising. This is in contradiction to the optimal advertising policy in the present paper.

Table 3 provides the comparative statics for α, β and $V_1(x)$ with respect to the parameters with proofs in Appendix B.

<Insert Table 3 here>

A comparison of the comparative statics in Table 2 and 3 shows the following main features. First, due to the additional complexity of the asymmetric case, there are a few more ambiguous effects. However, secondly, it appears that the change in own parameters has the same effect in the asymmetric case as a change in these parameters had for the symmetric case. This is to be expected since the first order effects likely dominate the second order effects, thus, yielding the same results as in the symmetric case. It becomes clear that a beneficial increase in own parameters (ρ_i, c_i, m_i, r_i) have a negative effect on the competitor's profits. Finally, the results for the amount of advertising u_i * are completely unambiguous and follow the same intuition as in the symmetric case. Note that the optimal advertising policy does not depend on the noisiness of the selling environment. This is a consequence of the linear form of the value function.

4.3. Characterization of the evolution path

We next examine the market share paths analytically. Inserting the values of the controls into the equations of motion (5), one obtains the following set of equations:

$$\begin{aligned} dx &= \left(\frac{\beta_1 \rho_1^{\,2}}{2c_1} + \delta - x\left(\frac{\beta_1 \rho_1^{\,2}}{2c_1} + \frac{\beta_2 \rho_2^{\,2}}{2c_2} + 2\delta \right) \right) dt + \sigma(x) dw, \quad x(0) = x_0, \\ dy &= \left(\frac{\beta_2 \rho_2^{\,2}}{2c_2} + \delta - y\left(\frac{\beta_1 \rho_1^{\,2}}{2c_1} + \frac{\beta_2 \rho_2^{\,2}}{2c_2} + 2\delta \right) \right) dt - \sigma(1-y) dw, \quad y(0) = 1 - x_0. \end{aligned} \tag{28}$$

These may be rewritten as stochastic integral equations

$$\begin{aligned} x(t) &= x_0 + \int_0^t \left(\frac{\beta_1 \rho_1^{\,2}}{2c_1} + \delta - x(s)\left(\frac{\beta_1 \rho_1^{\,2}}{2c_1} + \frac{\beta_2 \rho_2^{\,2}}{2c_2} + 2\delta \right) \right) ds + \int_0^t \sigma(x) dw, \\ y(t) &= (1 - x_0) + \int_0^t \left(\frac{\beta_2 \rho_2^{\,2}}{2c_2} + \delta - y(s)\left(\frac{\beta_1 \rho_1^{\,2}}{2c_1} + \frac{\beta_2 \rho_2^{\,2}}{2c_2} + 2\delta \right) \right) ds - \int_0^t \sigma(1-y) dw. \end{aligned} \tag{29}$$

The mean evolution path turns out to be independent of the nature of the stochastic disturbance. That is,

$$\begin{aligned} E[x(t)] &= x_0 + \int_0^t \left(\frac{\beta_1 \rho_1^{\,2}}{2c_1} + \delta - E[x(s)]\left(\frac{\beta_1 \rho_1^{\,2}}{2c_1} + \frac{\beta_2 \rho_2^{\,2}}{2c_2} + 2\delta \right) \right) ds, \\ E[y(t)] &= (1 - x_0) + \int_0^t \left(\frac{\beta_2 \rho_2^{\,2}}{2c_2} + \delta - E[y(s)]\left(\frac{\beta_1 \rho_1^{\,2}}{2c_1} + \frac{\beta_2 \rho_2^{\,2}}{2c_2} + 2\delta \right) \right) ds. \end{aligned} \tag{30}$$

These can be expressed as ordinary differential equations in $E[x(t)]$ and $E[y(t)]$ with the solutions given by

$$E[x(t)] = e^{-(\frac{\beta_1\rho_1^2}{2c_1}+\frac{\beta_2\rho_2^2}{2c_2}+2\delta)t} x_0 + (1-e^{-(\frac{\beta_1\rho_1^2}{2c_1}+\frac{\beta_2\rho_2^2}{2c_2}+2\delta)t}) \frac{\frac{\beta_1\rho_1^2}{2c_1}+\delta}{\frac{\beta_1\rho_1^2}{2c_1}+\frac{\beta_2\rho_2^2}{2c_2}+2\delta},$$

$$E[y(t)] = e^{-(\frac{\beta_1\rho_1^2}{2c_1}+\frac{\beta_2\rho_2^2}{2c_2}+2\delta)t} (1-x_0) + (1-e^{-(\frac{\beta_1\rho_1^2}{2c_1}+\frac{\beta_2\rho_2^2}{2c_2}+2\delta)t}) \frac{\frac{\beta_2\rho_2^2}{2c_2}+\delta}{\frac{\beta_1\rho_1^2}{2c_1}+\frac{\beta_2\rho_2^2}{2c_2}+2\delta}. \tag{31}$$

The long run equilibrium market shares $(\bar{x}, \bar{y})$ are obtained by taking the limit as $t \to \infty$ and are given by

$$\bar{x} = \frac{\frac{\beta_1\rho_1^2}{2c_1}+\delta}{\frac{\beta_1\rho_1^2}{2c_1}+\frac{\beta_2\rho_2^2}{2c_2}+2\delta}, \bar{y} = \frac{\frac{\beta_2\rho_2^2}{2c_2}+\delta}{\frac{\beta_1\rho_1^2}{2c_1}+\frac{\beta_2\rho_2^2}{2c_2}+2\delta}. \tag{32}$$

Thus, the expected market shares converge to the form resembling the attraction models commonly used in marketing. However, while an attraction model would rate the attractiveness of each firm based on its lower cost, higher productivity of advertising, and higher advertising, it would exclude exogenous market phenomena such as churn.

To further characterize the evolution path, we next calculate the variance of the market shares at each point in time. A specification of the disturbance function is required for this. We will use $\sigma(x)dw = \sigma\sqrt{x(1-x)}dw$, where σ is a positive constant and, recalling the discussion in Section 3 and equation (8), it can be seen that market shares will remain in $(0,1)$.

An application of Itô's formula to equation (28) provides the following result:

$$d(x(t)^2) = \left[2x\left(\frac{\beta_1\rho_1^2}{2c_1} + \delta - x(\frac{\beta_1\rho_1^2}{2c_1} + \frac{\beta_2\rho_2^2}{2c_2} + 2\delta) \right) + \sigma^2 x(1-x) \right] dt + 2x\sigma\sqrt{x(1-x)}dw \tag{33}$$

Rewriting this as a stochastic integral, taking the expected value, and rewriting as a differential equation, we get

$$\frac{dE[x(t)^2]}{dt}=(\frac{\beta_1\rho_1^{\,2}}{c_1}+2\delta+\sigma^2)E[x(t)]-(\frac{\beta_1\rho_1^{\,2}}{c_1}+\frac{\beta_2\rho_2^{\,2}}{c_2}+4\delta+\sigma^2)E[x(t)^2]. \quad (34)$$

Inserting the solution for $E[x(t)]$ from (31), we obtain a first order linear differential equation in the second moment $E[x(t)^2]$.

$$\frac{dE[x^2]}{dt}+(\frac{\beta_1\rho_1^{\,2}}{c_1}+\frac{\beta_2\rho_2^{\,2}}{c_2}+4\delta+\sigma^2)E[x^2]$$
$$=\frac{(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\delta)(\frac{\beta_1\rho_1^{\,2}}{c_1}+2\delta+\sigma^2)}{\frac{\beta_1\rho_1^{\,2}}{2c_1}+\frac{\beta_2\rho_2^{\,2}}{2c_2}+2\delta}+e^{-(\frac{\beta_1\rho_1^{2}}{2c_1}+\frac{\beta_2\rho_2^{2}}{2c_2}+2\delta)t}\left((\frac{\beta_1\rho_1^{\,2}}{c_1}+2\delta+\sigma^2)x_0-\frac{(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\delta)(\frac{\beta_1\rho_1^{\,2}}{c_1}+2\delta+\sigma^2)}{\frac{\beta_1\rho_1^{\,2}}{2c_1}+\frac{\beta_2\rho_2^{\,2}}{2c_2}+2\delta}\right)$$
(35)

The solution is

$$E[x(t)^2]=x_0^{\,2}e^{-2(\frac{\beta_1\rho_1^{2}}{2c_1}+\frac{\beta_2\rho_2^{2}}{2c_2}+2\delta+\frac{\sigma^2}{2})t}+\frac{(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\delta)(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\delta+\frac{\sigma^2}{2})}{(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\frac{\beta_2\rho_2^{\,2}}{2c_2}+2\delta+\frac{\sigma^2}{2})(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\frac{\beta_2\rho_2^{\,2}}{2c_2}+2\delta)}(1-e^{-2(\frac{\beta_1\rho_1^{2}}{2c_1}+\frac{\beta_2\rho_2^{2}}{2c_2}+2\delta+\frac{\sigma^2}{2})t})$$
$$+\frac{e^{-(\frac{\beta_1\rho_1^{2}}{2c_1}+\frac{\beta_2\rho_2^{2}}{2c_2}+2\delta)t}-e^{-2(\frac{\beta_1\rho_1^{2}}{2c_1}+\frac{\beta_2\rho_2^{2}}{2c_2}+2\delta+\frac{\sigma^2}{2})t}}{\frac{\beta_1\rho_1^{\,2}}{2c_1}+\frac{\beta_2\rho_2^{\,2}}{2c_2}+2\delta+\sigma^2}\left((\frac{\beta_1\rho_1^{\,2}}{c_1}+2\delta+\sigma^2)x_0-\frac{(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\delta)(\frac{\beta_1\rho_1^{\,2}}{c_1}+2\delta+\sigma^2)}{\frac{\beta_1\rho_1^{\,2}}{2c_1}+\frac{\beta_2\rho_2^{\,2}}{2c_2}+2\delta}\right).$$
(36)

We can calculate the convergence of the second moment, as the influence of the initial condition disappears. That is,

$$\lim_{t\to\infty}E[x(t)^2]=\frac{(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\delta)(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\delta+\frac{\sigma^2}{2})}{(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\frac{\beta_2\rho_2^{\,2}}{2c_2}+2\delta+\frac{\sigma^2}{2})(\frac{\beta_1\rho_1^{\,2}}{2c_1}+\frac{\beta_2\rho_2^{\,2}}{2c_2}+2\delta)}. \quad (37)$$

Written in this form, it becomes clear that when $\sigma = 0$ the expression is just $\bar{x}^2$ so that the variance is appropriately zero in the absence of stochastic effect. More generally, when $\sigma = 0$, $E[x(t)^2] = \left(E[x(t)]\right)^2$ holds for all t. For $\sigma > 0$ the standard deviation is $\sqrt{E[x(t)^2] - \left(E[x(t)]\right)^2}$.

Similar results are easily obtained for the second firm. We present the results for the mean and variance of the long-run market share in Proposition 2.

Proposition 2: For the advertising game described in (10):

(a) The mean market shares in the long run are given by (32),

$$\left(\bar{x} = \frac{\frac{\beta_1 \rho_1^2}{2c_1} + \delta}{\frac{\beta_1 \rho_1^2}{2c_1} + \frac{\beta_2 \rho_2^2}{2c_2} + 2\delta}, \bar{y} = \frac{\frac{\beta_2 \rho_2^2}{2c_2} + \delta}{\frac{\beta_1 \rho_1^2}{2c_1} + \frac{\beta_2 \rho_2^2}{2c_2} + 2\delta} \right).$$

(b) The variance of the market shares in the long run are obtained from (37) and (32) as $E[x(t)^2] - \left(E[x(t)]\right)^2$ and for both firms are given by

$$\frac{\left(\frac{\beta_1 \rho_1^2}{2c_1} + \delta\right)\left(\frac{\beta_2 \rho_2^2}{2c_2} + \delta\right)\sigma^2 / 2}{\left(\frac{\beta_1 \rho_1^2}{2c_1} + \frac{\beta_2 \rho_2^2}{2c_2} + 2\delta + \sigma^2 / 2\right)\left(\frac{\beta_1 \rho_1^2}{2c_1} + \frac{\beta_2 \rho_2^2}{2c_2} + 2\delta\right)^2}.$$

4.4. Illustration

Illustrative market shares may be obtained for different parameter values. We choose the parameter values $r = 0.05$, $\delta = 0.01$, symmetric margins $m_1 = m_2 = 1$, asymmetric firm strengths $R_1 = 1, R_2 = 4$ and an initial starting point at $x(0) = 0.5$. In practice, a decision calculus approach could be followed to obtain the parameter values. Using Mathematica, we find that the only real positive root for the quartic polynomial is $\beta_1 = 0.264545$ and the corresponding $\beta_2 = 0.43069$. Finally, we specify $\sigma(x)dw = \sigma\sqrt{x(1-x)}dw$, with $\sigma^2 = 0.5$, and use Microsoft Excel to plot equations (28) and (31).[3]

<Insert Figure 1 here>

Figure 1 shows a sample path. One can see that the path hovers around the mean. It never stays on the mean as it is continuously disrupted due to the Brownian motion. We can calculate a confidence interval if we assume that the path is approximately normally distributed around the mean. Then $E[x(t)] \pm 1.96\sqrt{E[x(t)^2] - \left(E[x(t)]\right)^2}$ provides the 95% confidence interval for the market share path. While we know that the distribution is not normal, nevertheless, as Figure 2 shows, the proposed confidence interval does an adequate job of tracking the market shares. Since the normal distribution is not bounded between zero and one, the confidence interval may exceed the minimum or maximum market share as happened in this case. In the next subsection, we will obtain the equilibrium distribution of the market share, which enables us to provide the exact confidence intervals for the equilibrium market share.

[3] To simulate a market share path, the procedure described in Zwillinger (1998, p.702, equation 182.3) was used; i.e., for the SDE $dx(t) = a(x)dt + b(x)dw(t)$, the numerical approximation is $x(t+\Delta) = x(t) + a(x(t))\Delta + b(x(t))\sqrt{\Delta}\varsigma(t)$. The $\{\varsigma(t)\}$ are i.i.d. Normal with mean 0 and variance 1 generated using Excel's random number generator (Tools -> Data Analysis -> Random Number Generation). The time step was $\Delta = 0.01$.

<Insert Figure 2 here>

This analysis has the value that it provides a diagnostic tool for management to handle market share fluctuations. Whereas minor fluctuations within the confidence bands may call for cursory examination, overstepping the bands signals the need for a detailed review. This is because it may be indicative of a shift in underlying market parameters and hence, requires a reevaluation of the advertising spending policies. Secondly, the market share fluctuations directly cause fluctuations in advertising spending according to Proposition 1 and, hence, one can simulate the advertising budget as well.

4.5. Probability distribution of market shares

We mentioned in the previous section that the probability densities of the market shares are not necessarily normally distributed. This raises the obvious question of whether the density functions can be determined explicitly, or at least approximated. We devote this section to examining this issue.

An important property of the solution $x(t)$ of an Itô stochastic differential equation

$$dx(t) = a(x,t)dt + b(x,t)dw(t), x(s) = z$$

is that it is a Markov process. The transition probability of this Markov process has a density $p(t,x;s,z)$ for going from market share z at time s to market share x at time $t > s$, that satisfies the Fokker-Planck equation

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}(ap) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(b^2 p) = 0, \ p(t,x;t,z) = \delta(x-z)$$

(also known as the Kolmogorov forward equation).

For our problem, we shall first obtain and then attempt to solve the Fokker-Planck equation. To keep the intermediate steps simple, we temporarily make use of the notation $A_1 = \frac{\beta_1 \rho_1^2}{2c_1} + \delta$ and $A_2 = \frac{\beta_2 \rho_2^2}{2c_2} + \delta$, and derive the results only for firm 1. Firm 1's stochastic differential equation, from equation (28), is

$$dx = (A_1 - (A_1 + A_2)x)dt + \sigma\sqrt{x(1-x)}dw, \quad x(0) = x_0 . \tag{38}$$

The corresponding Fokker-Planck equation is given by

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}((A_1 - (A_1 + A_2)x)p) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 x(1-x)p) = 0 , \tag{39}$$

which simplifies to

$$\frac{\partial p}{\partial t} + \frac{\sigma^2 x(x-1)}{2}\frac{\partial^2 p}{\partial x^2} + ((2\sigma^2 - (A_1 + A_2))x + A_1 - \sigma^2)\frac{\partial p}{\partial x} + (\sigma^2 - (A_1 + A_2))p = 0 . \tag{40}$$

This partial differential equation could not be explicitly solved. Nevertheless, we will attempt to find the density of the steady state market share by $\lim_{t\to\infty} p(t, x; s, z)$. Let $f(x)$ denote this density, since it can be shown to be independent of s, z and t. To recapitulate, what we started off wanting to know was the density $p(t, x; 0, x_0)$ of firm 1's market share at time t given that it starts at a point x_0 at time zero. By looking for the long-run stationary probability density of the market share, essentially we are willing to ignore the initial transient part of the solution. For density $f(x)$, we can set $\frac{\partial p}{\partial t} = 0$ in equation (40) and obtain the second order ordinary differential equation

$$\frac{\sigma^2 x(x-1)}{2}\frac{d^2 f}{dx^2} + ((2\sigma^2 - (A_1 + A_2))x + A_1 - \sigma^2)\frac{df}{dx} + (\sigma^2 - (A_1 + A_2))f = 0 . \tag{41}$$

A slight rearrangement of terms puts this in canonical form where it is identifiable as a Gaussian hypergeometric equation (Polyanin and Zaitsev 2003, p.234):

$$x\left(x-1\right)\frac{d^2 f}{dx^2}+\left(\left(4-\frac{2(A_1+A_2)}{\sigma^2}\right)x-\left(2-\frac{2A_1}{\sigma^2}\right)\right)\frac{df}{dx}+\left(2-\frac{2(A_1+A_2)}{\sigma^2}\right)f=0\,. \tag{42}$$

We obtain the solution from Polyanin and Zaitsev (2003, p.236, Table 17) to be

$$f(x)=x^{\frac{2A_1}{\sigma^2}-1}(1-x)^{\frac{2A_2}{\sigma^2}-1}\left(C_1+C_2\int x^{-\frac{2A_1}{\sigma^2}}(1-x)^{\frac{-2A_2}{\sigma^2}}dx\right). \tag{43}$$

To determine the constants of integration, we can employ the following two properties. First, the density should integrate to 1 and second, the expected value of the market share should be $\bar{x}$, which we have already calculated is equal to $A_1/(A_1+A_2)$.

After some reflection, we realize that we can always set $C_2=0$ because then $f(x)$ is recognizable as the density of a Beta distribution. The result is given in Proposition 3.

Proposition 3: The densities of the stationary distributions of the market shares are given by the Beta density as follows: For firm 1,

$$f(x)=\frac{\Gamma(\frac{2A_1}{\sigma^2}+\frac{2A_2}{\sigma^2})}{\Gamma(\frac{2A_1}{\sigma^2})\Gamma(\frac{2A_2}{\sigma^2})}x^{\frac{2A_1}{\sigma^2}-1}(1-x)^{\frac{2A_2}{\sigma^2}-1}, \tag{44}$$

where $\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}dx,\ s>0$ *is the gamma function. For firm 2, by symmetry, f(y) is obtained by interchanging x with y and* A_1 *with* A_2 *in (44).*

(Proof: The density integrates to one by definition, while the mean of the Beta distribution is given by $\frac{2A_1/\sigma^2}{2A_1/\sigma^2 + 2A_2/\sigma^2} = \frac{A_1}{A_1 + A_2}$ (Mood, et al. 1974). Thus, all the required conditions are satisfied.)

As a further check, the variance of the Beta distribution, given by $\frac{A_1 A_2 \sigma^2 / 2}{(A_1 + A_2 + \sigma^2/2)(A_1 + A_2)^2}$, matches the direct calculation of the variance of firm 1's market share.

To see how this may be applied, we now return to the illustrative example of section 4.4. Inserting parameter values $A_1 = 0.539$, $A_2 = 3.46$ and $\sigma^2 = 0.5$, the Beta density is

$$f(x) = \frac{\Gamma(16)}{\Gamma(2.156)\Gamma(13.84)} x^{1.156}(1-x)^{12.84} = 292.39 x^{1.156}(1-x)^{12.84}\,. \quad (45)$$

We can now compute the 95% confidence intervals:

$$\begin{aligned} &292.39\int_0^l \; x^{1.156}(1-x)^{12.84} = 0.025 \Rightarrow l = 0.02, \\ &292.39\int_0^m \; x^{1.156}(1-x)^{12.84}\,dx = 0.975 \Rightarrow m = 0.33. \end{aligned} \quad (46)$$

These provide a more accurate 95% confidence interval for the equilibrium market share of firm 1 than by assuming a normal distribution. These are sketched in Figure 2. The confidence intervals for firm 2 can be obtained in a similar manner.

5. Conclusions

We examined a dynamic duopoly with stochastic disturbances and employ closed-loop methods to solve the problem. The model was analyzed using stochastic differential game theory and explicit solutions were obtained. The effects of the several different parameters were discussed for symmetric and asymmetric firms.

The paper extends the work of Sethi (1983) to include competitive advertising response and the work of Sorger (1989) by including stochastic analysis and a churn term in the dynamics that is consistent with the original Vidale-Wolfe formulation and which ensures that in the absence of competitive advertising, market shares will converge. The effect of churn is not straightforward. That is, its effect can be decomposed into two parts; one is to reduce competition by making advertising less effective, hence causing a decrease in equilibrium advertising. the other is to disproportionately reduce share of the higher market share firm. Thus, higher churn benefits a firm with low market share but has ambiguous effects for a higher share firm.

A simple rule describes the optimal advertising expenditure, which is that it should be proportional to the square root of the opponent's market share (Proposition 1). In other words, when the market share of a firm is less, it is necessary to advertise more, and vice versa. A large portion of the discussion in Sections 4.1 and 4.2 is devoted to determining the proportionality constant, particularly its endogenous component β_i, and obtaining an analytical expression for it. While it is given by a simple expression when the firms are symmetric, unfortunately, it is not simple to state the proportionality constant for asymmetric firms. An explicit formula has been provided in the appendix, however. Furthermore, the dependence of β_i as well as the amount of advertising is provided by means of comparative statics. The comparative statics give directional adjustments to make to the amount of advertising in case of parameter changes.

In Section 4.3, we characterized the evolution path by using stochastic calculus to provide the mean and variance of the market shares. The former resembles an attraction model. In Section 4.5, we examined the probability distribution for the market shares by solving the Fokker-Planck equation in the limiting case and showing that it is the Beta distribution. The fact that these commonly used forms for market share emerge endogenously from the analysis additionally validates our modeling assumptions. An illustration demonstrated the usability of the analysis in terms of tracking the market shares (Section 4.4).

A few limitations and extensions should be mentioned. The present paper deals with a duopoly model of advertising competition. Duopoly models are of significant interest since they represent the advertising situation in many markets (Erickson 1992). Nevertheless, there are other markets characterized by three or more competitors. Extension of the present model and analysis along the lines of Erickson (1995, 2003) and Fruchter (1999), to an oligopoly is, therefore, important. Likewise, extending the model to incorporate additional decision variables such as price is important (Thompson and Teng 1984).

The comparative statics presented in the paper represent hypotheses for empirical testing. Lack of support for the hypotheses would indicate either the need to change modeling assumptions or that marketing managers are using suboptimal methods. However, whether discrepancies occur due to the validity of the modeling, or suboptimality of marketing practice, they are important to discover. Thus, empirical investigation would be fruitful.

Appendix A: Proof of Comparative Statics for Table 2

(a) We start by rewriting $r\beta = m - \frac{3\beta^2\rho^2}{4c} - 2\delta\beta$ from equation (16) as

$$G(\beta) \equiv m - \frac{3\beta^2\rho^2}{4c} - (r+2\delta)\beta = 0\,. \tag{A1}$$

Note that $\partial G/\partial\beta < 0$. Hence, for any parameter θ, the implicit function theorem

$\partial\beta/\partial\theta = -\frac{\partial G/\partial\theta}{\partial G/\partial\beta}$ implies that $sign(\partial\beta/\partial\theta) = sign(\partial G/\partial\theta)$. It follows that β

decreases when r, δ or ρ increase, and increases when m or c increase.

(b) For u_1*, it is helpful to write $u_1* = \beta\left(\frac{\rho\sqrt{1-x}}{2c}\right)$ and insert this in (A1) to get

$$G(u_1*) \equiv m - \frac{3cu_1*^2}{(1-x)} - (r+2\delta)\frac{2cu_1*}{\rho\sqrt{1-x}} = 0\,. \tag{A1}$$

Comparative statics with respect to r, δ and m are the same as for β. However, comparative statics for ρ and c are reversed.

(c) We express α as $\alpha = \frac{1}{r}\left(\frac{\beta^2\rho^2}{4c} + \beta\delta\right)$. Comparative statics with respect to ρ,

c, m and r are clearly the same as for β. Only the effect of an increase in δ on α needs careful calculation. This is done as follows:

$$\begin{gathered}\alpha = \frac{(r-\delta)(-6R\beta)+6Rm}{18Rr} = \frac{(\delta-r)\beta+m}{3r} \\ sign\left(\frac{\partial\alpha}{\partial\delta}\right) = sign\left(\beta + (\delta-r)\frac{\partial\beta}{\partial\delta}\right)\end{gathered} \tag{A3}$$

We calculate $\frac{\partial\beta}{\partial\delta}=\frac{\frac{2W*2}{2\sqrt{W^2+12Rm}}-2}{6R}=\frac{-2\beta}{\sqrt{W^2+12Rm}}$, and insert it into the above equation to continue. Thus,

$$\begin{aligned}sign\left(\frac{\partial\alpha}{\partial\delta}\right)&=sign\left(\beta-\frac{2\beta(\delta-r)}{\sqrt{W^2+12Rm}}\right)\\&=sign\left(\sqrt{W^2+12Rm}-2(\delta-r)\right)=sign\left(6R\beta+3r\right),\end{aligned}\tag{A4}$$

which is positive.

(d) Since $V_1(x)=\alpha+\beta x$, whenever comparative statics for α and β are in the same direction, $V_1(x)$ also has identical comparative statics. Thus, we need only to calculate the comparative statics with respect to δ. Let us observe that

$$\begin{aligned}&V_i=\alpha+\beta x=\frac{1}{18Rr}\left[(\delta-r+3rx)\left(\sqrt{W^2+12Rm}-W\right)+6Rm\right]\\&\Rightarrow sign\frac{\partial V_i}{\partial\delta}=sign\left[\left(\sqrt{W^2+12Rm}-W\right)+\frac{2(\delta-r+3rx)}{\sqrt{W^2+12Rm}}\left(W-\sqrt{W^2+12Rm}\right)\right]\\&=sign\left[1-\frac{2(\delta-r+3rx)}{\sqrt{W^2+12Rm}}\right]=sign\left[\sqrt{W^2+12Rm}-W+3r(1-2x)\right]\\&=sign\left[\frac{\sqrt{(r+2\delta)^2+12Rm}-(r+2\delta)}{6r}+\frac{1}{2}-x\right].\end{aligned}\tag{A5}$$

If $x\le 1/2$, the sign is positive. If $x>1/2+\frac{\sqrt{(r+2\delta)^2+12Rm}-(r+2\delta)}{6r}$, the sign is negative.

Appendix B: Proof of Uniqueness of Solution

We want to show that there exists a unique solution to the differential game. This implies showing that there exists a unique (β_1, β_2) that satisfies

$$r_1\beta_1 = m_1 - \frac{\beta_1^{\,2}\rho_1^{\,2}}{4c_1} - \frac{\beta_1\beta_2\rho_2^{\,2}}{2c_2} - 2\beta_1\delta, \tag{B1}$$

$$r_2\beta_2 = m_2 - \frac{\beta_2^{\,2}\rho_2^{\,2}}{4c_2} - \frac{\beta_1\beta_2\rho_1^{\,2}}{2c_1} - 2\beta_2\delta \tag{B2}$$

$$\beta_1 > 0 \text{ and } \beta_2 > 0\,. \tag{B3}$$

We begin by reducing (B1) and (B2) to a quartic equation in β_1,

$$\begin{gathered}3R_1^{\,2}\beta_1^{\,4} + 2R_1(W_1+W_2)\beta_1^{\,3} + (4R_2m_2 - 2R_1m_1 - W_1^{\,2} + 2W_1W_2)\beta_1^{\,2} \\ + 2m_1(W_1-W_2)\beta_1 - m_1^{\,2} = 0.\end{gathered} \tag{B4}$$

This may be rewritten in its simplest form as

$$F(\beta_1) \equiv \beta_1^{\,4} + \kappa_1\beta_1^{\,3} + \kappa_2\beta_1^{\,2} + \kappa_3\beta_1 - \kappa_4 = 0\,, \tag{B5}$$

where,

$$\kappa_1 = \frac{2(W_1+W_2)}{3R_1}, \kappa_2 = \frac{\left(4m_2R_2 - 2m_1R_1 - W_1^{\,2} + 2W_1W_2\right)}{3R_1^{\,2}}, \kappa_3 = \frac{2m_1\left(W_1 - W_2\right)}{3R_1^{\,2}}, \kappa_4 = \frac{m_1^{\,2}}{3R_1^{\,2}}\,. \tag{B6}$$

Every quartic equation has four roots. Excluding the fortuitous cases where two or more roots are equal, the following results are easily observed.

1. When $\beta_1 \to \pm\infty$, $F(\beta_1) \to \infty$ and when $\beta_1 = 0$, $F(\beta_1) < 0$. Since $F(\beta_1)$ is differentiable, it is continuous. Thus, it must cross the x-axis at least twice ensuring at least one positive and one negative real root. If there are only two real

roots, one will be positive and one negative. If all four roots are real, they will be either three positive and one negative, or three negative and one positive.

2. In the case of three real positive roots, ordering them from the smallest to the largest, the slope at the second largest root must be negative. To see this, write $F(\beta_1) = (\beta_1 - \beta_1(1))(\beta_1 - \beta_1(2))(\beta_1 - \beta_1(3))(\beta_1 - \beta_1(4))$, where $\beta_1(1) < 0$, $\beta_1(2) > 0$, $\beta_1(3) > \beta_1(2)$, $\beta_1(4) > \beta_1(3)$ are the four roots. Then, the slope at $\beta_1(3)$, $F'(\beta_1)\big|_{\beta_1=\beta_1(3)} = (\beta_1(3) - \beta_1(1))(\beta_1(3) - \beta_1(2))(\beta_1(3) - \beta_1(4))$, is negative.

3. We can calculate the slope $F'(\beta_1)$ directly from equation (B4) and evaluate it at any positive real root. The following steps differentiate (B4) and then reapply equation (B4) to obtain a simpler expression:

$$
\begin{aligned}
& F'(\beta_1)\big|_{\beta_1=F^{-1}(0)} \\
&= 12R_1^2\beta_1^3 + 6R_1(W_1+W_2)\beta_1^2 + 2(4R_2m_2 - 2R_1m_1 - W_1^2 + 2W_1W_2)\beta_1 + 2m_1(W_1 - W_2) \\
&= \frac{2}{\beta_1}[6R_1^2\beta_1^4 + 3R_1(W_1+W_2)\beta_1^3 + (4R_2m_2 - 2R_1m_1 - W_1^2 + 2W_1W_2)\beta_1^2 + 2m_1(W_1 - W_2)\beta_1 \\
&= \frac{2}{\beta_1}[3R_1^2\beta_1^4 + R_1(W_1+W_2)\beta_1^3 + m_1W_2\beta_1 + m_1(m_1 - W_1\beta_1)] > 0
\end{aligned}
\tag{B7}
$$

The last expression is positive since from (B3), $m_1 > \beta_1 W_1$.

It follows from points 2 and 3 above that there is only one real positive root and, hence, a unique solution to the differential game.

To obtain an explicit solution, we utilize the Mathematica 4.1 software to generate four solutions to (B5):

$$
\beta_1(1) = -\frac{\kappa_1}{4} - \frac{g}{2} - \frac{1}{2}\sqrt{\frac{3\kappa_1^2}{4} - 2\kappa_2 - g^2 - \frac{-\kappa_1^3 + 4\kappa_1\kappa_2 - 8\kappa_3}{4g}} \tag{B8}
$$

$$\beta_1(2) = -\frac{\kappa_1}{4} - \frac{g}{2} + \frac{1}{2}\sqrt{\frac{3\kappa_1^2}{4} - 2\kappa_2 - g^2 - \frac{-\kappa_1^3 + 4\kappa_1\kappa_2 - 8\kappa_3}{4g}} \tag{B9}$$

$$\beta_1(3) = -\frac{\kappa_1}{4} + \frac{g}{2} - \frac{1}{2}\sqrt{\frac{3\kappa_1^2}{4} - 2\kappa_2 - g^2 + \frac{-\kappa_1^3 + 4\kappa_1\kappa_2 - 8\kappa_3}{4g}} \tag{B10}$$

$$\beta_1(4) = -\frac{\kappa_1}{4} + \frac{g}{2} + \frac{1}{2}\sqrt{\frac{3\kappa_1^2}{4} - 2\kappa_2 - g^2 + \frac{-\kappa_1^3 + 4\kappa_1\kappa_2 - 8\kappa_3}{4g}} \tag{B11}$$

Where two intermediate terms g and h are defined below.

$$g \equiv \sqrt{\frac{\kappa_1^2}{4} - \frac{2\kappa_2}{3} + \frac{2^{1/3}(\kappa_2^2 - 3\kappa_1\kappa_3 - 12\kappa_4)}{3h} + \frac{h}{32^{1/3}}} \tag{B12}$$

$$h \equiv \left(2\kappa_2^3 - 9\kappa_1\kappa_2\kappa_3 + 27\kappa_3^2 - 27\kappa_1^2\kappa_4 + 72\kappa_2\kappa_4 + \sqrt{-4(\kappa_2^2 - 3\kappa_1\kappa_3 - 12\kappa_4)^3 + (2\kappa_2^3 - 9\kappa_1\kappa_2\kappa_3 + 27\kappa_3^2 - 27\kappa_1^2\kappa_4 + 72\kappa_2\kappa_4)^2}\right)^{1/3} \tag{B13}$$

We pick β_1 as the only real positive solution out of the four roots, i.e,

$$\beta_1 = \beta_1(i^*), \text{ where } i^* = \{i \in \{1,2,3,4\} \mid \beta_1(i) > 0\}. \tag{B14}$$

While i^* may depend on the data, there will only be one i^* in every case.

Appendix C: Proof of Comparative Statics for Table 3

(a) To obtain comparative statics for β_i, we define

$$G_1(\beta_1,\beta_2) \equiv m_1 - \frac{\beta_1^{\,2}\rho_1^{\,2}}{4c_1} - \frac{\beta_1\beta_2\rho_2^{\,2}}{2c_2} - \beta_1(r_1+2\delta) = 0,$$
$$G_2(\beta_1,\beta_2) \equiv m_2 - \frac{\beta_2^{\,2}\rho_2^{\,2}}{4c_2} - \frac{\beta_1\beta_2\rho_1^{\,2}}{2c_1} - \beta_2(r_2+2\delta) = 0. \tag{C1}$$

Then, for any parameter θ, we use the implicit function theorem (Simon and Blume 1994, p.354):

$$\begin{pmatrix} \dfrac{\partial\beta_1}{\partial\theta} \\ \dfrac{\partial\beta_2}{\partial\theta} \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial G_1}{\partial\beta_1} & \dfrac{\partial G_1}{\partial\beta_2} \\ \dfrac{\partial G_2}{\partial\beta_1} & \dfrac{\partial G_2}{\partial\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{\partial G_1}{\partial\theta} \\ \dfrac{\partial G_2}{\partial\theta} \end{pmatrix}. \tag{C2}$$

With some calculation, this can be written as

$$\begin{pmatrix} \dfrac{\partial\beta_1}{\partial\theta} \\ \dfrac{\partial\beta_2}{\partial\theta} \end{pmatrix} = -\frac{1}{\Delta}\begin{pmatrix} -(\dfrac{\beta_2\rho_2^{\,2}}{2c_2} + \dfrac{\beta_1\rho_1^{\,2}}{2c_1} + r_2 + 2\delta) & \dfrac{\beta_1\rho_2^{\,2}}{2c_2} \\ \dfrac{\beta_2\rho_1^{\,2}}{2c_1} & -(\dfrac{\beta_1\rho_1^{\,2}}{2c_1} + \dfrac{\beta_2\rho_2^{\,2}}{2c_2} + r_1 + 2\delta) \end{pmatrix} \begin{pmatrix} \dfrac{\partial G_1}{\partial\theta} \\ \dfrac{\partial G_2}{\partial\theta} \end{pmatrix}, \tag{C3}$$

where, $\Delta > 0$. It can be shown in a straightforward manner that

$$\frac{\partial\beta_1}{\partial c_2} > 0, \frac{\partial\beta_1}{\partial m_1} > 0, \frac{\partial\beta_1}{\partial m_2} < 0, \frac{\partial\beta_1}{\partial r_1} < 0, \frac{\partial\beta_1}{\partial r_2} > 0, \frac{\partial\beta_1}{\partial\rho_2} < 0, \frac{\partial\beta_1}{\partial\delta} < 0\,. \tag{C4}$$

However, the cases for c_1 and ρ_1 are ambiguous:

$$sign(\frac{\partial\beta_1}{\partial c_1}) = sign[\frac{\beta_1\rho_1^{\,2}}{2c_1} - \frac{\beta_2\rho_2^{\,2}}{2c_2} + r_2 + 2\delta],$$
$$sign(\frac{\partial\beta_1}{\partial\rho_1}) = sign[-(\frac{\beta_1\rho_1^{\,2}}{2c_1} - \frac{\beta_2\rho_2^{\,2}}{2c_2} + r_2 + 2\delta)]. \tag{C5}$$

(b) For comparative statics for u_1*, we insert $u_1* = \dfrac{\beta_1\rho_1\sqrt{1-x}}{2c_1}, u_2* = \dfrac{\beta_2\rho_2\sqrt{x}}{2c_1}$

into (C1) to obtain

$$\begin{aligned} G_1(u_1*,u_2*) &\equiv m_1 - \frac{c_1u_1*^2}{1-x} - \frac{2c_1u_1*u_2*\rho_2}{\rho_1\sqrt{1-x}\sqrt{x}} - \frac{2c_1u_1*(r_1+2\delta)}{\rho_1\sqrt{1-x}} = 0, \\ G_2(u_1*,u_2*) &\equiv m_2 - \frac{c_2u_2*^2}{x} - \frac{2c_2u_1*u_2*\rho_1}{\rho_2\sqrt{1-x}\sqrt{x}} - \frac{2c_2u_2*(r_2+2\delta)}{\rho_2\sqrt{x}} = 0. \end{aligned} \tag{C6}$$

Then, the implicit function theorem can be written as

$$\begin{pmatrix} \dfrac{\partial u_1*}{\partial\theta} \\ \dfrac{\partial u_2*}{\partial\theta} \end{pmatrix} = -\frac{1}{\Delta}\begin{pmatrix} -(\dfrac{2c_2u_2}{x} + \dfrac{2c_2u_1\rho_1}{\rho_2\sqrt{1-x}\sqrt{x}} + \dfrac{2c_2(r_2+2\delta)}{\rho_2\sqrt{x}}) & \dfrac{2c_1u_1\rho_2}{\rho_1\sqrt{1-x}\sqrt{x}} \\ \dfrac{2c_2u_2\rho_1}{\rho_2\sqrt{1-x}\sqrt{x}} & -(\dfrac{2c_1u_1}{1-x} + \dfrac{2c_1u_2\rho_2}{\rho_1\sqrt{1-x}\sqrt{x}} + \dfrac{2c_1(r_1+2\delta)}{\rho_1\sqrt{1-x}}) \end{pmatrix}\begin{pmatrix} \dfrac{\partial G_1}{\partial\theta} \\ \dfrac{\partial G_2}{\partial\theta} \end{pmatrix}, \tag{C7}$$

where $\Delta > 0$. The calculations are straightforward, and we provide only the results here:

$$\frac{\partial u_1*}{\partial c_1} < 0, \frac{\partial u_1*}{\partial c_2} > 0, \frac{\partial u_1*}{\partial m_1} > 0, \frac{\partial u_1*}{\partial m_2} < 0, \frac{\partial u_1*}{\partial r_1} < 0, \frac{\partial u_1*}{\partial r_2} > 0, \frac{\partial u_1*}{\partial \rho_1} > 0, \frac{\partial u_1*}{\partial \rho_2} < 0, \frac{\partial u_1*}{\partial \delta} < 0. \tag{C8}$$

(c) For α_1, we note from equation (25) that in many cases α_1 will have the same comparative statics as β_1. These relationships are as follows:

$$\frac{\partial\alpha_1}{\partial c_2} > 0, \frac{\partial\alpha_1}{\partial m_1} > 0, \frac{\partial\alpha_1}{\partial m_2} < 0, \frac{\partial\alpha_1}{\partial r_1} < 0, \frac{\partial\alpha_1}{\partial r_2} > 0, \frac{\partial\alpha_1}{\partial \rho_2} < 0. \tag{C9}$$

The cases for c_1, ρ_1 and δ are unclear.

(d) The unambiguous results for V_1 occur when comparative statics for α_1 and β_1 are in the same direction, which is true for all parameters except c_1, ρ_1 and δ.

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Table 1: Notation

$x(t) \in [0,1]$	Market share for firm 1, $x(0) = x_0$.
$y(t) = 1 - x(t)$	Market share for firm 2, $y(0) = 1 - x_0$.
$u_i(x(t), y(t), t) \geq 0$	Advertising rate by firm i at time t.
$\rho_i > 0$	Advertising effectiveness parameter for firm i.
$\delta > 0$	Market share decay or churn parameter.
$r_i > 0$	Discount rate for firm i.
$C(u_i(t))$	Cost of advertising, parameterized as $c_i u_i(t)^2, c_i > 0$.
$\sigma(x(t), y(t))dw(t)$	Disturbance function with standard white noise.
V_i	Value function for firm i.
α_i, β_i	Components of the value function.
$R_i \equiv \rho_i^2 / 4c_i, W_i \equiv r_i + 2\delta$, $A_i \equiv \frac{\beta_i \rho_i^2}{2c_i} + \delta$	Some useful intermediate terms.

Table 2: Comparative Statics with Symmetric Firms (Proofs in Appendix A)

Variables	Parameters				
Note, $R=\frac{\rho^2}{4c}, W=r+2\delta$.	c	ρ	m	δ	r
$\alpha=\frac{(r-\delta)\left(W-\sqrt{W^2+12Rm}\right)+6Rm}{18Rr}$	+	-	+	+	-
$\beta=\frac{\sqrt{W^2+12Rm}-W}{6R}$	+	-	+	-	-
$u_1{*}=\frac{(\sqrt{W^2+12Rm}-W)\rho\sqrt{1-x}}{12Rc}$	-	+	+	-	-
Value function, $V_1(x)=\alpha+\beta x$	+	-	+	?	-

Legend: increase (+), decrease (-), ambiguous (?)

Table 3: Comparative Statics with Asymmetric Firms (Proofs in Appendix C)

Variables	Parameters				
	c_i, c_j	ρ_i, ρ_j	m_i, m_j	δ	r_i, r_j
α_i	? , +	? , -	+ , -	?	- , +
β_i	? , +	? , -	+ , -	-	- , +
u_i *	- , +	+ , -	+ , -	-	- , +
$V_i(x)$	? , +	? , -	+ , -	?	- , +

Legend: increase (+), decrease (-), ambiguous (?)

Fig. 1. Market Share Trajectories given Optimal Advertising Decisions

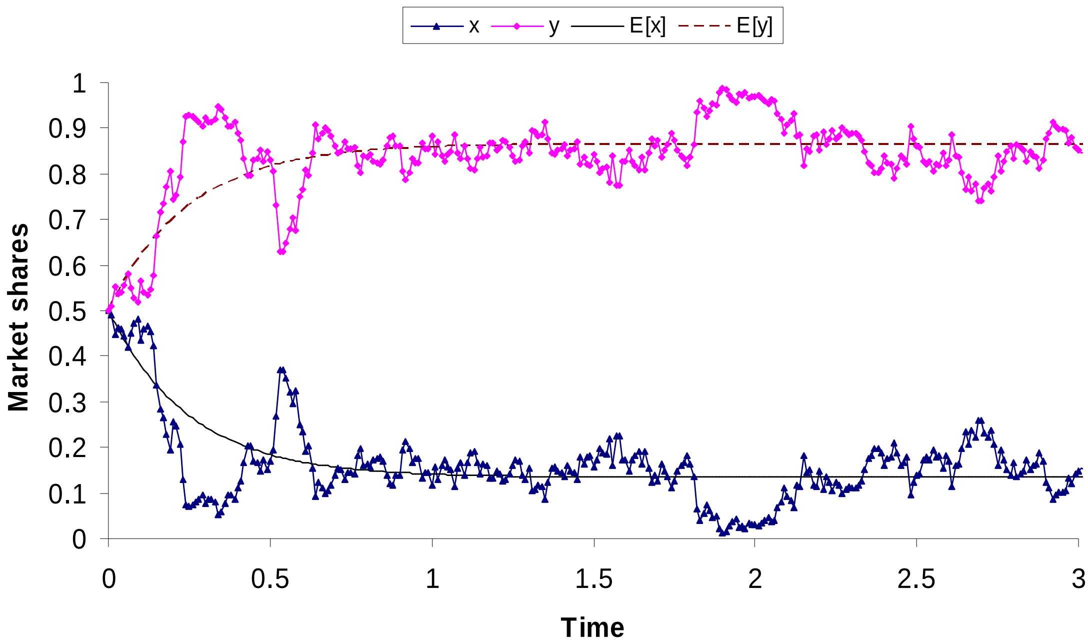

Fig. 2. Market Share for Firm 1, Normal density 95% Confidence Interval (dashed lines), and Equilibrium Market Share 95% Confidence Interval (dotted lines)