Co-op Advertising in Dynamic Retail Oligopolies

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Decison Sciences, forthcoming

July 2011

ABSTRACT

We study a supply chain in which a consumer goods manufacturer sells its product through a retailer. The retailer undertakes promotional expenditures, such as advertising, to increase sales and to compete against other retailer(s). The manufacturer supports the retailer's promotional expenditure through a cooperative advertising program by reimbursing a portion (called the subsidy rate) of the retailer's promotional expenditure. To determine the subsidy rate, we formulate a Stackelberg differential game between the manufacturer and the retailer, and a Nash differential subgame between the retailer and the competing retailer(s). We derive the optimal feedback promotional expenditures of the retailers and the optimal feedback subsidy rate of the manufacturer, and show how they are influenced by market parameters. An important finding is that the manufacturer should support its retailer only when a subsidy threshold is crossed. The impact of competition on this threshold is non-monotone. Specifically, the manufacturer offers more support when its retailer competes with one other retailer but its support starts decreasing with the presence of additional retailers. In the case where the manufacturer sells through all retailers, we show under certain assumptions that it should support only one dominant retailer. We also describe how we can incorporate retail price competition into the model.

***Subject Areas**: **Co-op Advertising, Marketing channels, Stackelberg Differential Games, Subsidy Rate**.*

INTRODUCTION

Manufacturers often delegate local advertising and promotion of their products to their retail partners when the latter understand local market conditions better and can promote the product more effectively. However, though a retailer will advertise the manufacturer's product in its own interests, to increase sales and profitability, it may not do so to the extent desired by the manufacturer. Therefore, additional incentives are needed to align the interests of the retailer and the manufacturer. This paper examines such an incentive, which is the use of cooperative advertising programs by consumer goods manufacturers to support retailers' promotional efforts. In these programs, manufacturers offer to partially reimburse retailers for their promotional expenditure on the manufacturers' product in the retailers' local market (Bergen & John, 1997). According to the 2008 Co-op Advertising Programs Sourcebook, about $50 billion of media advertising was financed through cooperative advertising programs. Dant and Berger (1996) reported that 25-40% of all manufacturers use co-op advertising.

The percentage of retail promotional expenditure that is reimbursed is called the *subsidy rate*. For example, Nature's Bounty paid 50% of the advertising cost through its joint cooperative advertising program. Dutta, Bergen, John, and Rao (1995) report the subsidy rate figures for other well-known manufacturers as well as industry averages. In particular, for consumer goods, the average subsidy rate was 74.44%, which comes from the subsidy rates of 88.38% for consumer convenience products (e.g., books, milk, bread, toothpaste, health aids) and 68.95% for consumer non-convenience products (e.g. women's apparel, tires).

For the manufacturer, it is important to correctly determine what the subsidy rate should be and whether a cooperative advertising program should be offered at all. While some previous research has examined this issue in a bilateral monopoly setting (e.g., He, Prasad, & Sethi, 2009), the situation is more complicated if the retailer is in a competitive market. Nevertheless, duopolistic or oligopolistic competition is quite common in many industries and this element will be examined in this paper.

A feature of market share response involving advertising is that it often has a dynamic behavior.

This means that the market share can change over time due to current advertising expenditures as well as the carry-over effect of past advertising. The advertising rate decisions are functions of time as well. In addition, advertising should compensate for forgetting or churn for a more realistic representation, which is consistent with the recent literature. Due to the dynamics, the modeling procedure uses dynamic counterparts of the Stackelberg and simultaneous Nash games.

We consider a focal supply chain consisting of a manufacturer and a retailer of a nondurable consumer good. In addition, there are competing outside retailers, so called because they are not selling the manufacturer's product or being supported by the manufacturer. Within this setup, the leader-follower structure between the manufacturer and the retailers is formulated as a Stackelberg differential game. At the retail level, each retailer's market share depends on its own and its competitors' current and past advertising expenditures. Retailers are allowed to be strategic, meaning that they take each others' actions and reactions into consideration when making decisions. The retailers simultaneously choose their advertising levels. Competition between them is formulated as a Nash differential game with dynamics that follow those in papers such as Sorger (1989), Prasad and Sethi (2004), and Prasad, Sethi, and Naik (2009).

As extensions, we will consider the case when the manufacturer sells through all of the retailers, and therefore has an option of supporting some or all of them. When the retailers are symmetric and only one retailer provides a higher margin with the others giving equal margins, we show that the manufacturer will support only the dominant retailer. We also extend the model to incorporate retail price competition.

Preview of Main Results

This paper addresses the following issues relating to the supply chain with retail competition: What are the optimal advertising expenditures for the retailers over time? What is the optimal subsidy rate for the manufacturer over time, and how is it influenced by firm and market parameters? What is the impact of competition (i.e., market structure) on the optimal subsidy rate?

Some results from the analysis of the model are: First, the retailers' optimal advertising is a

function of market shares and the value functions of the retailers and the manufacturer are linear in their market shares. Second, the manufacturer should support its retailer when a set of conditions—called the subsidy threshold—on the profit margin, advertising effectiveness, and other parameters is exceeded. The optimal subsidy rate, not the subsidy amount, is a fixed percentage, independent of market share. Third, we find that the impact of competition on the manufacturer's subsidy threshold is non-monotone. That is, when the retailer is in a duopoly market, the manufacturer offers a higher level of support to its retailer, and over a wider range of parameters, in comparison to the results obtained in a monopoly retail market. However, as the number of competitors increases, the manufacturer offers a lower level of support to its retailer and does so over a narrower range of parameters in an oligopoly market compared to that in a duopoly retail market.

Literature Review

This paper focuses on the optimal advertising decisions in a decentralized supply chain in a dynamic environment. There are a few static models of cooperative advertising in the marketing literature (Berger, 1972; Berger & Magliozzi, 1992; Dant & Berger, 1996; Bergen & John, 1997; Kali, 1998; Kim & Staelin, 1999; Huang, Li, & Mahajan, 2002). Among these, Dant and Berger (1996) study the role of cooperative advertising in franchising systems. Bergen and John (1997) explore the impact of advertising spillover and manufacturer and retailer differentiation on the subsidy rate.

There also exists a stream of both empirical and analytical research that examines the impact of advertising on market share dynamics. Sethi (1977), Feichtinger, Hartl, and Sethi (1994) and He, Prasad, Sethi, and Gutierrez (2007) provide surveys of dynamic advertising models. Most directly related to our paper are Jørgensen, Sigue, and Zaccour (2000), Jørgensen, Taboubi, and Zaccour (2001, 2003), Jørgensen and Zaccour (2003), Karray and Zaccour (2005), and He et al. (2009). All of these papers model the Stackelberg game in a single-manufacturer, single-retailer channel. In support of the analytical models, there are empirical studies that use advertising as an explanatory variable in the market share dynamics. For example, Chintagunta and Jain (1995) consider

data on the Pepsi-Coke duopoly in soft drinks (1976-1989 data), the P&G vs. Unilever rivalry in detergents (1978-1992 data), and the Anheuser-Busch vs. Miller rivalry in beer (1974-1989 data). The market share in the cola market was similarly studied in the models of advertising competition by Chintagunta and Vilcassim (1992), Erickson (1992), and Fruchter and Kalish (1997). Erickson (2009a) studies a triopoly involving Anheuser-Busch, SAB Miller, and Molson Coors. Erickson (2009b) tested advertising competition in an oligopoly using data from 1995-2006 on Coke, Sprite, Pepsi, Mountain Dew, and Dr. Pepper. More recently, Naik, Prasad, and Sethi (2008) looked at the data relating advertising to awareness share for five Italian automobile brands.

This paper is related to research by He et al. (2009, 2011) who also study co-op advertising policies in a dynamic supply chain. However, He et al. (2009) do not model retail competition. He et al. (2011) extend He et al. (2009) by modeling retail competition, but consider only a duopoly retail market with two symmetric retailers. We provide a generalization of these models by examining the oligopoly case and retailer asymmetry. As to the contrast in results, while each paper explains why manufacturers sometimes offer and sometimes do not offer co-op support, the threshold conditions for offering support are distinct. He et al. (2009) show that when the discount rate and the decay rate are small, the manufacturer provides support for the retailer when its margin is larger than the retailer's margin, while He et al. (2011) show that in a duopoly retail market, the manufacturer supports the retailer when its margin is two-thirds of the retailer's margin. Our results generalize the effects of margins and competition, showing that co-op support can either increase or decrease with increasing retail competitors.

The paper is organized as follows. In the next section, we formulate a model in a duopoly retail market with asymmetric retailers and the related assumptions. Next, we present the analysis and results. Thereafter, we study an oligopoly market and evaluate the impact of competition on the subsidy rate. The last section concludes with a summary and directions for future research. Proofs are included in Appendix A and Appendix B provides two extensions: the manufacturer selling through all retailers and the incorporation of retail price competition.

MODEL

As shown in Figure 1, we consider the distribution channel for a consumer good with a manufacturer and a retailer, Retailer 1, who competes with an outside retailer, Retailer 2. Competitive efforts could include circulars, TV and radio spots, and other forms of local advertising. We later extend the model to study competitive markets in which Retailer 1 competes with $n-1$ outside retailers.

Insert Figure 1 here.

Let $x(t)$ denote the market share of Retailer 1 at time $t \geq 0$ which depends on its own and its competitor's advertising efforts. Thus, the market share of Retailer 2 at time t is $1-x(t)$. The manufacturer supports Retailer 1's advertising activities by sharing a portion of retail advertising expenditures. This support for Retailer 1, that is, the subsidy rate at time t, is denoted by $\theta(t)$. Table 1 summarizes the notation in the paper.

Insert Table 1 here.

The advertising expenditure is assumed to be quadratic in the advertising effort $u_i(t)$, $i = 1,2$, so that the manufacturer's and Retailer i's advertising expenditure rates at time t are therefore $\theta(t)u_1^2(t)$, and $(1-\theta(t))\,u_1^2(t)$ and $u_2^2(t)$, respectively. The quadratic cost assumption is common in the literature (Deal, 1979; Sorger, 1989; Chintagunta and Jain, 1992; Prasad and Sethi, 2004; Bass, Krishnamoorthy, Prasad, and Sethi, 2005; and He et al., 2009). It implies diminishing returns to advertising expenditure.

The sequence of events in the game is as follows: The manufacturer sets the subsidy rate policy for Retailer 1. Then, the retailers choose their respective advertising efforts. Sales are then realized.

The market share dynamics of Retailer 1 is given by

$$\dot{x}(t) = \rho_1 u_1(t)\sqrt{1-x(t)} - \rho_2 u_2(t)\sqrt{x(t)} - \delta_1 x(t) + \delta_2(1-x(t)), \quad x(0) = x_0 \in [0,1], \quad (1)$$

where the advertising response constants ρ_1 and ρ_2 determine each retailer's advertising effectiveness, and the market share decay constants δ_1 and δ_2 determine the rate at which consumers churn. This competitive extension of the Sethi (1983) model has been used in models by Sorger (1989), Prasad and Sethi (2004) and Bass et. al (2005), and validated in empirical studies by Chintagunta and Jain (1992) and Naik et al. (2008). It has the desirable properties that market share is non-decreasing in own advertising and non-increasing in the competitor's advertising, with concave response and saturation.

In the retail competition stage, after a feedback subsidy policy $\theta(x)$ has been announced by the manufacturer, each retailer maximizes its discounted profit stream with respect to its advertising decision and subject to the market share dynamics. The retailers' objectives are

$$V_1(x_0) = \max_{u_1(t),\, t\geq 0} \int_0^\infty e^{-rt}\left(m_1 x(t) - (1-\theta(x(t)))\, u_1^2(t)\right) dt, \quad (2)$$

$$V_2(x_0) = \max_{u_2(t),\, t\geq 0} \int_0^\infty e^{-rt}\left(m_2(1-x(t)) - u_2^2(t)\right) dt, \quad (3)$$

where m_i is Retailer i's margin for $i \in \{1,2\}$ and r is the discount rate. We use time as the argument for advertising and subsidy rate decisions and, with a slight abuse of notation, market share as the argument for the feedback solutions for these decisions. This will be explained presently.

In the above equations, if we denote the value function of Retailer i by $V_i(x)$ for any market share x, then $V_i(x_0)$ denotes the optimal value of Retailer i's discounted total profit at time zero. Solving the differential game in Equations (1)-(3) yields Retailer i's feedback advertising effort $u_i(x|\theta(x))$ in response to the manufacturer's announced subsidy rate policy $\theta(x)$.

Moving to the first stage of the game, the manufacturer anticipates the retailers' reaction functions when solving for its subsidy rate. Therefore, the manufacturer's problem is given by

$$V(x_0) \quad = \quad \max_{\theta(t),\, t\geq 0} \int_0^\infty e^{-rt} \left(Mx(t) - \theta(t)(u_1(x(t)|\theta(t)))^2 \right) dt, \tag{4}$$

subject to

$$\begin{aligned} \dot{x}(t) \quad = \quad & \rho_1 u_1(x(t)|\theta(t))\sqrt{1-x(t)} - \rho_2 u_2(x(t)|\theta(t))\sqrt{x(t)} \\ & -\delta_1 x(t) + \delta_2(1-x(t)), \quad x(0) = x_0 \in [0,1], \end{aligned} \tag{5}$$

where M is the margin on the manufacturer's sales to Retailer 1.

Solution of the optimal control problem (Equations (4)-(5)) requires us to obtain the optimal subsidy rate $\theta^*(t)$, $t \geq 0$. This can be accomplished here by obtaining the optimal subsidy policy in feedback form, which we will, as mentioned earlier, express as $\theta^*(x)$. From this, we obtain $\theta^*(t) = \theta^*(x^*(t))$, $t \geq 0$, along the optimal path $x^*(t)$, $t \geq 0$. Furthermore, we can express Retailer i's feedback advertising effort, again with an abuse of notation, as $u_i^*(x) = u_i^*(x|\theta^*(x))$. Note that the policies $\theta^*(x)$ and $u_i^*(x)$, $i = 1,2$, constitute a feedback Stackelberg equilibrium of the problem shown in Equations (2)-(5), which is time consistent, as opposed to the open-loop Stackelberg equilibrium, which, in general, is not. Substituting these policies into Equation (1) yields the market share process $x^*(t)$, $t \geq 0$, and the respective decisions $\theta^*(x^*(t))$ and $u_i^*(x^*(t))$ at time $t \geq 0$.

Upon examining the Stackelberg differential game described in Equations (2)-(5), we can observe that if $\delta_2 = \rho_2 = 0$, then the problem for the focal supply chain is reduced to one with no retail competition since Retailer 2 is competitively ineffectual. In this case, the model reduces to that of He et al. (2009) as a special case. We omit the repetition of their results and focus on the general case.

ANALYSIS AND RESULTS

In this section, we attempt to find the optimal solutions to the basic setup. We are interested in whether and when the manufacturer should support Retailer 1. We find that the manufacturer supports Retailer 1 only when a threshold is exceeded. When the manufacturer does support the retailer, we find that the optimal subsidy policy is easy to implement as it is constant over time.

Proposition 1: The feedback Stackelberg equilibrium of the game presented in Equations (2)-(5) is characterized as follows:

(a) The optimal advertising decisions of the retailers are given by

$$u_1(x|\theta) = \frac{V_1^{'}\rho_1\sqrt{1-x}}{2(1-\theta)}, \quad u_2(x|\theta) = -\frac{V_2^{'}\rho_2\sqrt{x}}{2}. \tag{6}$$

(b) The optimal subsidy rate of the manufacturer has the form

$$\theta(x) = \left(\frac{2V^{'}(x) - V_1^{'}(x)}{2V^{'}(x) + V_1^{'}(x)}\right)^{+}. \tag{7}$$

(c) The value functions V_1, V_2, and V for Retailer 1, Retailer 2, and the manufacturer, respectively, satisfy the three simultaneous differential equations (A3), (A4), and (A7) in the Appendix.

Consistent with earlier studies, the optimal advertising expenditure of each retailer is proportional to its uncaptured market share. Thus, when the market shares are lower, the retailers should advertise more heavily. We also see that for any given subsidy rate $\theta \geq 0$, Retailer 1's optimal advertising effort is increasing in θ. Reasonably, the more supportive the manufacturer, the higher the promotional effort of Retailer 1. On the other hand, Retailer 2's best response is independent of θ.

At this point, as in Sethi (1983) and He et al. (2009), we look for a solution in linear value functions which work for such formulations because of the square-root feature in the dynamics Equation (5). Specifically, we set $V_1 = \alpha_1 + \beta_1 x$, $V_2 = \alpha_2 + \beta_2(1-x)$, and $V = \alpha_M + \beta_M x$ in

Equations (A3)-(A4) and (A7), where the unknown parameters α_1, β_1, α_2, β_2, α_M, and β_M are constants. Then, by equating the coefficients of x on both sides of Equations (A8)-(A10), we get six simultaneous algebraic equations that can indeed be solved to obtain the six unknown parameters. These results are summarized in Proposition 2.

Proposition 2:

(a) The retailers' optimal advertising decisions are given by

$$u_1^*(x) = \frac{\beta_1 \rho_1 \sqrt{1-x}}{2(1-(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1})^+)}, \quad u_2^*(x) = \frac{\beta_2 \rho_2 \sqrt{x}}{2}. \tag{8}$$

(b) The optimal subsidy rate of the manufacturer is a constant given by

$$\theta^*(x) = \left(\frac{2\beta_M - \beta_1}{2\beta_M + \beta_1}\right)^+. \tag{9}$$

(c) The value functions of the two retailers and the manufacturer are linear in market share, i.e., $V_1 = \alpha_1 + \beta_1 x$, $V_2 = \alpha_2 + \beta_2 (1-x)$, and $V = \alpha_M + \beta_M x$, where the parameters α_1, β_1, α_2, β_2, α_M, and β_M are obtained by solving the system of Equations (A11)-(A16).

We note that the manufacturer's optimal subsidy rate is constant although it was allowed to vary with time and market share. However, since the optimal advertising rate varies with the market share, the total subsidy amount varies with the market share, and hence with time. From the manufacturer's persecutive, this policy is easy to implement because it does not require the manufacturer to continuously monitor market share.

There are now two cases to consider: $\theta^* = 0$ and $\theta^* > 0$.

Case $\theta^* = 0$

Whenever this case arises, and later in Proposition 3 we will determine the precise conditions on the parameters when it does, we must have the value-function coefficients that satisfy the condition

$\frac{2\beta_M-\beta_1}{2\beta_m+\beta_1} \leq 0$, and these coefficients must solve the system of equations obtained from Equations (A11)-(A16) by setting $\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+ = 0$.

As shown in Appendix A, Equations (A18) and (A20) can be used to obtain a quartic equation in β_1, for which there is a unique positive solution. That solution can be substituted into Equation (A18) to yield the solution of β_2, and then into Equation (A22) to obtain the solution of β_M.

Given these solutions, Proposition 3 presents the *subsidy threshold*, the point at which the manufacturer moves from $\theta^* = 0$ to $\theta^* > 0$.

Proposition 3: The subsidy threshold S is given by

$$S = 2M - m_1 - \frac{\beta_1^2\rho_1^2}{4}, \tag{10}$$

where β_1 is the unique positive solution of

$$\beta_1^4 + \kappa_1\beta_1^3 + \kappa_2\beta_1^2 - \kappa_3 = 0, \tag{11}$$

with $\kappa_1 = \frac{16(r+\delta_1+\delta_2)}{3\rho_1^4}$, $\kappa_2 = \frac{16(r+\delta_1+\delta_2)^2-8m_1\rho_1^2+16m_2\rho_2^2}{3\rho_1^4}$, and $\kappa_3 = \frac{16m_1^2}{3\rho_1^4}$. The manufacturer chooses $\theta^* > 0$ when $S > 0$ and $\theta^* = 0$ when $S \leq 0$. Furthermore, $\partial S/\partial M = 2 > 0$ at $S \leq 0$.

Note that S is computed by solving the system of equations (A17)-(A22) for $\theta^* = 0$, in which case $S \leq 0$. From the result $\partial S/\partial M > 0$ at $S = 0$, we can immediately see that when $S = 0$, a decrease in M will keep $\theta^* = 0$, whereas an increase in M will make it optimal for the manufacturer to set $\theta^* > 0$. This is because a higher margin from Retailer 1 gives the manufacturer a greater incentive to provide advertising support to Retailer 1.

Next, we state Corollary 1, which provides the comparative statics for symmetric retailers (i.e., $\delta_i = \delta,\ \rho_i = \rho,\ m_i = m,\ \alpha_i = \alpha$, and $\beta_i = \beta,\ i = 1,2$).

Corollary 1:

(a) We have the following comparative statics results for S w.r.t. the model parameters: $\partial S/\partial M > 0$, $\partial S/\partial m < 0$, $\partial S/\partial \rho < 0$, $\partial S/\partial r > 0$, $\partial S/\partial \delta > 0$.

(b) At the manifold $S = 0$, we have the following comparative statics results for the model parameters: $\partial M/\partial m > 0$, $\partial \rho/\partial m < 0$, $\partial M/\partial \delta < 0$, $\partial m/\partial \delta > 0$, $\partial \rho/\partial \delta > 0$, $\partial r/\partial \delta < 0$.

The comparative statics in Corollary 1(a) denote the impact of the model parameters on the subsidy threshold S. The increase of S with respect to M at $S = 0$ repeats our finding in the asymmetric case as reported in Proposition 3. Corollary 1(a) also states that the higher the retailers' margin m is, the lower the manufacturer's incentive to provide advertising support will be. Since investments in advertising have a greater impact on a retailer's profit when margins are high, a high-margin retailer has its own incentives to invest in advertising. This diminishes the need for the manufacturer to provide advertising support. A similar argument can be made for the decrease of S at $S \leq 0$ with the retailers' advertising effectiveness ρ. An increase in the discount rate r or the decay rate δ, on the other hand, makes it more difficult for a retailer to increase its market share through advertising. This prompts the manufacturer to provide more advertising support. Consequently, S increases with r and δ at $S \leq 0$.

The comparative statics in Corollary 1(b) represent the change needed in one model parameter to offset a change in another model parameter so as to maintain $\theta^* = 0$. First, consider $\partial M/\partial m$. We know from Corollary 1(a) that the subsidy threshold S decreases with m. Note also from Corollary 1(a) that S increases with M. Therefore, if m increases (decreases), then M must increase (decrease) to keep $S = 0$. If m decreases and M does not, then θ^* will become strictly positive. Similar effects can be noted for the impact of δ on m and ρ by looking at the effect of δ, m, and ρ on S in Corollary 1(a). For $\partial M/\partial \delta$, the signs of $\partial S/\partial M$ and $\partial S/\partial \delta$ in Corollary 1(a) can be used to show that if δ increases (decreases), then M must decrease (increase) to maintain $S = 0$. Next, consider the interaction between m and ρ. Since S decreases with both m and ρ, an increase in m has to be accompanied by a decrease in ρ in order to keep $S = 0$. For $\partial r/\partial \delta$, note that δ works like a discount rate, as is known in the economics literature. Here, specifically, if r increases by ε, then δ must decrease by 2ε to keep $S = 0$.

He et al. (2011) show that when r and δ are small compared to m and M, the subsidy rate $\theta^* > 0$ if $M > 2m/3$. In other words, the manufacturer provides advertising support to Retailer 1

operating in a symmetric duopoly if its margin from Retailer 1 exceeds $2m/3$. Stated differently, if m increases by ε, then M must increase by $2\varepsilon/3$ to maintain $S = 0$. Let us note that this result follows also as a corollary of Proposition 3 when the retailers are symmetric and $r = \delta = 0$.

Case $\theta^* > 0$

When the solution results in $S > 0$, we know that $\theta^* > 0$. Then, by substituting $\left(\frac{2\beta_M - \beta_1}{2\beta_M + \beta_1}\right)^+ = \frac{2\beta_M - \beta_1}{2\beta_M + \beta_1}$ into Equations (A11)-(A16), we have the following system of equations to solve for the value-function coefficients:

$$
\begin{aligned}
r\alpha_1 &= \frac{1}{8}\beta_1\rho_1^2(2\beta_M + \beta_1) + \beta_1\delta_2, && (12)\\
r\beta_1 &= m_1 - \frac{1}{8}\beta_1\rho_1^2(2\beta_M + \beta_1) - \frac{1}{2}\beta_1\beta_2\rho_2^2 - \beta_1(\delta_1 + \delta_2), && (13)\\
r\alpha_2 &= \frac{1}{4}\beta_2^2\rho_2^2 + \beta_2\delta_1, && (14)\\
r\beta_2 &= m_2 - \frac{1}{4}\beta_2\rho_1^2(2\beta_M + \beta_1) - \frac{1}{4}\beta_2^2\rho_2^2 - \beta_2(\delta_1 + \delta_2), && (15)\\
r\alpha_M &= \frac{1}{16}\rho_1^2(2\beta_M + \beta_1)^2 + \beta_M\delta_2, && (16)\\
r\beta_M &= M - \frac{1}{16}\rho_1^2(2\beta_M + \beta_1)^2 - \frac{1}{2}\beta_2\beta_M\rho_2^2 - \beta_M(\delta_1 + \delta_2). && (17)
\end{aligned}
$$

This requires numerical analysis. Figure 2 illustrates the effect of the retailer's margin on the manufacturer's subsidy rate for the following sample parameter values: $r = 0.03$, $\rho_1 = \rho_2 = 0.5$, $\delta_1 = 0.07$, $\delta_2 = 0.1$, $M = 0.5$. Furthermore, for the effect of Retailer 1's (Retailer 2's) margin, we set $m_2 = 0.5$ ($m_1 = 0.5$).

Insert Figure 2 here.

We already know from Corollary 1(a) that $\partial S/\partial m < 0$ at $S \leq 0$ in the symmetric case. The result shown in Figure 2—that θ^* decreases, and at a decreasing rate, as the margin m_1 increases—is consistent with our finding in the symmetric case and represents its generalization to the asymmetric

case. The reason is that with low margin m_1, Retailer 1 will under-advertise and the manufacturer's profit will suffer on account of decreased sales. So, then, it is in the interest of the manufacturer to provide a subsidy to encourage the retailer to advertise more. As m_1 increases, Retailer 1 has its own incentive to advertise, and therefore the manufacturer does not need to offer as much in the way of a subsidy. As m_1 keeps increasing, the manufacturer ceases to participate altogether. We see from Figure 2 that θ^* indeed becomes zero at $m_1 \approx 0.8$, where the switch from "cooperative advertising" to "no cooperative advertising" takes place. This generalizes the result obtained in He et al. (2009) in the absence of competition, to the competitive environment.

The effect of the margin of the non-supported retailer (i.e. Retailer 2) on the offer of cooperative advertising to Retailer 1, appears to be weak. Note that at $m_1 = 0.5$, the two curves in Figure 2 cross. At this point, as m_1 increases, *ceteris paribus*, θ^* decreases sharply whereas as m_2 increases, θ^* increases slowly. This means that as m_2 increases, Retailer 1 faces greater advertising competition from the competing retailer, and this induces the manufacturer to support Retailer 1 at a slightly higher rate. On the other hand, as m_1 increases, as has already been mentioned, it increases the incentive of Retailer 1 to advertise of its own accord and thus the manufacturer reduces its support significantly. Even though m_2 has a weak effect on θ^*, it may still make a difference between the manufacturer supporting or not supporting Retailer 1. Numerical analysis shows that when $m_1 = m_2 = 0.5$, then $\theta^* = 0.278$. However, as m_2 decreases from 0.5 (e.g., $m_2 = 0.2$) the manufacturer stops offering advertising support to Retailer 1. On the other hand, as m_2 increases from 0.5 to 0.8, θ^* increases only marginally from a support of 27.8% to 29.1%.

Insert Figure 3 here.

In Figure 3, the fixed parameter values are: $m_1 = 0.2$, $m_2 = 0.5$, $r = 0.03$, $\delta_1 = 0.07$, $\delta_2 = 0.1$, $M = 0.5$. To find the effect of Retailer 1's (Retailer 2's) advertising effectiveness, we set $\rho_2 = 0.5$ ($\rho_1 = 0.5$). Consistent with the comparative statics result for the symmetric case in Corollary 1(a) (that $\partial S/\partial \rho < 0$), this figure shows that as the advertising effectiveness of Retailer 1 increases, the

degree of support by the manufacturer diminishes. The reason is that given the greater effectiveness of advertising, the retailer has an incentive to advertise at a higher level even without the support of the manufacturer. Retailer 2's advertising effectiveness does not have a great impact, but as it increases it slowly raises the subsidy rate.

Insert Figure 4 here.

Finally, Figure 4 shows that as the sum of the market share decay rates for the two retailers increases, the subsidy rate to Retailer 1 increases. For this figure, the remaining parameter values are: $m_1 = 0.5$, $m_2 = 0.2$, $r = 0.03$, $\rho_1 = 0.5$, $\rho_2 = 0.5$, $M = 0.8$. Note that we have plotted θ^* against the sum of the decay rates since it is clear from the dynamics in Equation (1) that the decay rate for Retailer 1 is $(\delta_1 + \delta_2)$. It is for this reason that we can easily see from Equations (12)-(17) that β_1, β_2, and β_M, and, therefore, θ^*, are affected by the sum $(\delta_1 + \delta_2)$ and not by the individual decay rates. The result can be explained intuitively since the increase of $(\delta_1 + \delta_2)$ is similar to increasing the speed of the treadmill with which the advertising must keep up. Thus, as $(\delta_1 + \delta_2)$ increases, Retailer 1 finds it more expensive to maintain its market share and the manufacturer must offer a higher subsidy rate to Retailer 1 to adequately promote the product. This finding generalizes to the asymmetric case the analytical result obtained for the symmetric case in Corollary 1(a), that $\partial S/\partial \delta > 0$. Finally, note that the effect of the decay rates is most pronounced at their lower values and the decay rates do not have much effect on the subsidy rate at higher values.

OLIGOPOLY RETAIL MARKET

We are interested in examining the impact of retail competition on the optimal subsidy rate. To that end, we consider an oligopoly in which Retailer 1 has $n-1$ outside competitors. For tractability, we focus on the case of n symmetric retailers, where $m_i = m$, $\rho_i = \rho$, $\delta_i = \delta$ for $i \in \{1, 2, ...n\}$. As before, the manufacturer offers a cooperative advertising subsidy only to Retailer 1.

The demand dynamics generalizes the duopoly specification as follows:

$$\dot{x}_i(t) = \frac{n}{n-1}\rho u_i\sqrt{1-x_i} - \frac{1}{n-1}\sum_{j=1}^{n}\rho u_j\sqrt{1-x_j} - \delta\left(x_i - \frac{1}{n}\right), \quad x_i(0) = x_{i0}, \quad i \in \{1,2,...,n\}. \tag{18}$$

The first term captures the gain in market share from the competitors, the second term the loss of market share to the competitors, and the third term the market share churn. This model of retail competition is discussed and validated in Prasad et al. (2009). It should be noted that with more than three retailer types there is the possibility that market shares can go out-of-bounds, referring to Naik et al. (2008) for this type of dynamics. By assuming that the competing retailers are all homogenous with each other, though not necessarily with the focal retailer, we side-step the out-of-bounds problem while allowing for any number of retailers.

Since $\sum_{i=1}^{n} x_i = 1$, the differential game is properly formulated with $n-1$ independent states, which we choose to be $x_2, x_3, ..., x_n$. Each retailer maximizes its profit with respect to its advertising decision given the subsidy rate $\theta(x_2, x_3, ..., x_n)$ announced by the manufacturer. The objective functions of the n retailers are given by

$$V_1(x_{20}, x_{30}, ..., x_{n0}) = \max_{u_1(t),\, t\geq 0} \int_0^\infty e^{-rt}\left(m(1-\sum_{i=2}^{n} x_i(t)) - ((1-\theta(x_2(t), x_3(t), ..., x_n(t)))\, u_1^2(t)\right) dt, \tag{19}$$

$$V_i(x_{20}, x_{30}, ..., x_{n0}) = \max_{u_i(t),\, t\geq 0} \int_0^\infty e^{-rt}\left(m x_i(t) - u_i^2(t)\right) dt,\ i \in \{2,3,...,n\}, \tag{20}$$

subject to Equation (18).

The manufacturer's objective function, with a slight abuse of notation, is

$$V(x_{20}, x_{30}, ..., x_{n0}) = \max_{\theta(t),\, t\geq 0} \int_0^\infty e^{-rt}\left(M(1-\sum_{i=2}^{n} x_i(t)) - \theta(t) u_1^2(t)\right) dt, \tag{21}$$

subject to Equation (18).

Proposition 4: The feedback Stackelberg equilibrium with n symmetric retailers is characterized as follows:

(a) The optimal advertising decisions of the retailers are given by

$$u_1(x_2,x_3,...,x_n|\theta) = -\frac{\rho\sqrt{\sum_{j=2}^{n} x_j}}{2(n-1)(1-\theta)}\left(\sum_{k=2}^{n}\frac{\partial V_1}{\partial x_k}\right), \tag{22}$$

$$u_i(x_2,x_3,...,x_n|\theta) = \frac{\rho\sqrt{1-x_i}}{2(n-1)}\left(n\frac{\partial V_i}{\partial x_i} - \sum_{k=2}^{n}\frac{\partial V_i}{\partial x_k}\right), i \in \{2,3,...,n\}. \tag{23}$$

(b) The optimal subsidy rate of the manufacturer has the form

$$\theta(x_2,x_3,...,x_n) = \left(\frac{2B-A}{2B+A}\right)^{+}, \tag{24}$$

where $A = \sum_{j=2}^{n}\frac{\partial V_1}{\partial x_j}$ and $B = \sum_{j=2}^{n}\frac{\partial V}{\partial x_j}$.

(c) The value functions V_i for Retailer i, $i = 1,2,...,n$, and V for the manufacturer, satisfy the $n+1$ Equations (A31)-(A33) in Appendix A.

As before, we look for the linear value functions V_i, $i \in \{1,2,...,n\}$. To obtain their forms, we proceed as follows. Note that when $\theta = 0$, the n retailers simply play a Nash game. Since they are symmetric when $\theta = 0$, it is convenient to visualize the linear value function forms in terms of x_1, x_2, ..., x_n. Thus,

$$\begin{aligned}
V_1 &= \alpha + \beta x_1 + \gamma x_2 + \gamma x_3 + ... + \gamma x_n,\\
V_2 &= \alpha + \gamma x_1 + \beta x_2 + \gamma x_3 + ... + \gamma x_n,\\
V_3 &= \alpha + \gamma x_1 + \gamma x_2 + \beta x_3 + ... + \gamma x_n,\\
&...\\
V_n &= \alpha + \gamma x_1 + \gamma x_2 + \gamma x_3 + ... + \beta x_n.
\end{aligned}$$

Note that the above specification can have non-unique coefficients on account of a redundant state,

but they would express the same value function. For example, we can write V_1 in the following two ways:

$$\begin{aligned} V_1 &= \alpha + \frac{\beta}{2}x_1 + \frac{\beta}{2}\left(1 - \sum_{i=2}^{n} x_i\right) + \gamma x_2 + \gamma x_3 + \ldots + \gamma x_n \\ &= \left(\alpha + \frac{\beta}{2}\right) + \frac{\beta}{2}x_1 + \left(\gamma - \frac{\beta}{2}\right)x_2 + \left(\gamma - \frac{\beta}{2}\right)x_3 + \ldots + \left(\gamma - \frac{\beta}{2}\right)x_n. \end{aligned}$$

In order to obtain the value functions in terms of x_2, x_3, ..., x_n, we replace x_1 by $(1 - \sum_{i=2}^{n} x_i)$ to obtain

$$\begin{aligned} V_1 &= (\alpha + \beta) - (\beta - \gamma)\sum_{i=2}^{n} x_i, \\ V_2 &= (\alpha + \gamma) + (\beta - \gamma)x_2, \\ V_3 &= (\alpha + \gamma) + (\beta - \gamma)x_3, \\ &\ldots \\ V_n &= (\alpha + \gamma) + (\beta - \gamma)x_n. \end{aligned}$$

Let $\alpha + \beta = a$ and $\beta - \gamma = b$. Then, $\alpha + \gamma = a - b$, and we have

$$V_1 = a - b\sum_{j=2}^{n} x_j, \quad V_i = (a - b) + bx_i, \quad i = 2, 3, \ldots n. \tag{25}$$

We should note that the form of Equation (25) will have unique coefficients.

Similarly, for the case when $\theta > 0$, we can write the value functions for the manufacturer and the retailers as: $V = P - Q(\sum_{i=2}^{n} x_i)$, $V_1 = a_1 - b_1(\sum_{i=2}^{n} x_i)$, $V_i = (a - b) + bx_i$, $i = 2, 3, \ldots, n$, where a, b, a_1, b_1, P, and Q are constants. Note that when $\theta > 0$, Retailers i, for $i = 2, 3, \ldots, n$, are symmetric but Retailer 1 changes when it gets a subsidy. Therefore, the coefficients $a_1 \neq a$ and $b_1 \neq b$ when $\theta > 0$. The results of the case when $\theta > 0$ are summarized in Proposition 4.

Proposition 5:

(a) The retailers' optimal advertising decisions are given by

$$u_1^*(x_2,x_3,...,x_n) = \frac{b_1\rho\sqrt{\sum_{j=2}^n x_j}}{2(1-\theta)}, \quad u_i^*(x_2,x_3,...,x_n) = \frac{b\rho\sqrt{1-x_i}}{2}, \quad i \in \{2,3,...,n\}. \tag{26}$$

(b) The optimal subsidy rate of the manufacturer is a constant given by

$$\theta^*(x_2,x_3,...,x_n) = \left(\frac{2Q-b_1}{2Q+b_1}\right)^+. \tag{27}$$

(c) The value functions of the manufacturer and the retailers are linear in market share, i.e., $V = P - Q(\sum_{i=2}^n x_i)$, $V_1 = a_1 - b_1(\sum_{i=2}^n x_i)$, $V_i = (a-b) + bx_i$, $i = 2,3,...,n$, where the parameters a, b, a_1, b_1, P, and Q are obtained from the system of equations in (A37)-(A40) and (A42)-(A43).

Consistent with results in the duopoly retail market, the optimal advertising effort of each retailer is proportional to the square root of its uncaptured market share. Only Retailer 1's advertising effort is a function of the subsidy rate θ while, for the other retailers, the effort is independent of θ. The value-function parameters a, b, P, and Q for $\theta^* = 0$ are summarized in Corollary 2.

Corollary 2:

(a) The value-function coefficients of the retailers in the no-subsidy case are given by

$$\begin{aligned} a = \; & \frac{2(n-1)(2nr+(n-1)\delta)\sqrt{(n-1)^2(r+\delta)^2+m(n^2-1)\rho^2}}{n(n+1)^2 r\rho^2} \\ & -\frac{2(n-1)^2(r+\delta)(2nr+(n-1)\delta)+m(n-3)n(n+1)\rho^2}{n(n+1)^2 r\rho^2} \end{aligned} \tag{28}$$

$$b = \frac{2\sqrt{(n-1)^2(r+\delta)^2+m(n^2-1)\rho^2}-2(n-1)(r+\delta)}{(n+1)\rho^2}. \tag{29}$$

(b) The value-function coefficients of the manufacturer in the no-subsidy case are given by

$$P = \frac{2M\left(4\delta - 3\left(\sqrt{(r+\delta)^2 + 2m\rho^2} - r - \delta\right)\right)}{3\left(r + \delta + 3\sqrt{(r+\delta)^2 + 2m\rho^2}\right)} \tag{30}$$

$$Q = \frac{M(n^2-1)}{(n-1)(r+\delta) + n\sqrt{(n-1)^2(r+\delta)^2 + m(n^2-1)\rho^2}}. \tag{31}$$

The subsidy threshold S is presented in Proposition 6.

Proposition 6: The subsidy threshold S in the case of n symmetric retailers is given by

$$S = \frac{2Q}{b} - 1, \tag{32}$$

where b and Q are given by Equations (29) and (31), respectively.

When $r = \delta = 0$, the subsidy threshold in Equation (32) reduces to a simpler form as shown below.

Corollary 3: When $r = \delta = 0$, the expression for S in Equation (32) reduces to

$$S = \frac{(n+1)M - nm}{nm}. \tag{33}$$

From Corollary 3, we can see that $\theta > 0$ if $M > (\frac{n}{n+1})m$. That is, the manufacturer provides cooperative advertising support to Retailer 1 if its margin M is greater than $\frac{n}{n+1}$ times the retailer's margin m. We now compare this result to those in the cases of a duopoly retail market and a monopolist retailer.

Comparison of the Three Market Structures: Monopoly, Duopoly, and Oligopoly

We now examine the role of competition by comparing the optimal subsidy threshold in the retail duopoly and oligopoly cases to those in He et al. (2009) for a retail monopolist. He et al. (2009) show that when r and δ are small, the manufacturer subsidizes the retailer when $M > m$. Without a subsidy, the convex advertising cost and the shrinking market potential dissuade the monopolist retailer from advertising enough to capture the market potential that is beneficial to the manufacturer.

In a retail duopoly, the manufacturer subsidizes Retailer 1 when $M > 2m/3$. For n retailers, the manufacturer subsidizes Retailer 1 when $M > (\frac{n}{n+1})m$. In the limit, as n approaches ∞, the previous inequality reverts back to $M > m$, which is the same as that for a retail monopolist.

In other words, compared to the case of a monopolist retailer, the manufacturer provides support under a wider range of conditions when its retailer competes in a duopoly. But the range of support decreases monotonically (while always remaining higher than in the monopoly case) as the number of competitors increases from 2 to ∞, converging to the monopoly result as $n \to \infty$.

The numerical results suggest that this pattern of responses to the number of competitors, increasing and then retreating to the monopoly baseline, does not occur just for the subsidy threshold, but for the subsidy rate as well, including over the range of parameter values with $r > 0$ and $\delta > 0$. However, this pattern could only be proved for $r = \delta = 0$. For example, Figure 5 provides subsidy rates in monopoly, duopoly, and oligopoly as a function of the manufacturer's margin M. The fixed parameter values are $m = 0.5$, $r = 0.03$, $\rho = 0.5$, and $\delta = 0.1$. Furthermore, θ^*_{NC} denotes the optimal subsidy rate in the absence of retail competition, θ^*_D denotes the optimal subsidy rate in a retail duopoly, and θ^*_T that in a retail triopoly.

Insert Figure 5 here.

From the figure, the threshold value of M at which the manufacturer supports its retailer in the presence of competition is lower than that without competition. In particular, the manufacturer begins setting $\theta_D^* > 0$ when M increases to about 0.3 and $\theta_T^* > 0$ when M increases to about 0.33, compared to $\theta_{NC}^* > 0$ when $M \approx 0.38$. In other words, the manufacturer provides cooperative advertising support for a wider range of model parameters in the duopoly case than in the monopoly case, and the support range in the triopoly case lies between the cases of monopoly and duopoly. As is clear from Figure 5, a wider range of support is also associated with a higher level of support for every value of M larger than the threshold value where subsidy becomes positive. Finally, we see from the figure that while the subsidy rate under competition is higher than that in the absence of competition, the difference in subsidy rate declines as M increases.

The intuitive interpretation is as follows. First, when we move from a monopoly to a duopoly, Retailer 1 finds itself in competition with Retailer 2. Since Retailer 2's advertising effort is directed to wrest market share from Retailer 1, thereby hurting the manufacturer's profit, the manufacturer's support for Retailer 1 is higher in the presence of competing Retailer 2 than in its absence.

This implies that there is an effect, which we refer to as the primary effect, whereby the manufacturer will increase support so that its retailer will win or retain some share in a competitive market. However, if this was the only effect, we would expect the support to keep increasing as more competitors entered the market. But as the number of competitors increases, an interesting reversal occurs. Let us consider a triopoly with an additional Retailer 3. Now we find that the manufacturer's support to its retailer in a triopoly continues to be higher than that in monopoly, but lower than that in duopoly. This suggests a secondary effect of competitive entry.

The reason for this behavior appears to be that the competition between outside Retailers 2 and 3 creates an added incentive for Retailer 1 to advertise on its own accord, so the manufacturer does not need to support Retailer 1 in a triopoly as much as it would in a duopoly. But this secondary effect of free-riding does not dominate the primary effect, so as a result the triopoly support is still greater than in a monopoly case. Moreover, as n increases, the secondary effect increases monotonically so that support continues to decrease. However, support remains higher than in a

monopoly as indicated by the fact that the support threshold converges to the monopoly level as $n \to \infty$.

Before concluding this section, we should recapitulate that those effects that were seen in the absence of competition also hold under competition. First, the result that the subsidy rate declines with higher retailer margin continues to hold under competition, and the intuition is that less incentive is needed since the retailer's interest in increasing market share is more closely aligned with that of the manufacturer. Second, we find that as Retailer 2's margin increases, the manufacturer offers more support to Retailer 1. The intuition behind this is that because Retailer 2 with a higher margin is a fiercer competitor, the manufacturer's support for Retailer 1 is higher to encourage that retailer to advertise more to counter the effect of the competition.

CONCLUSIONS

In this paper, we consider a consumer goods manufacturer who sells through a retailer in competition with outside retailers. The retailers invest in local advertising effort, while the manufacturer decides whether or not to support its retailer's advertising activities by providing a subsidy rate. We model the interaction between the manufacturer and its retailer as a Stackelberg differential game and the interaction between the competing retailers as a Nash differential game.

We derive the optimal feedback advertising effort levels of the retailers and the feedback subsidy rate of the manufacturer. Compared to He et al. (2009), who do not model retail competition, we find that in a retail duopoly the presence of a competing retailer induces the manufacturer to provide a higher level of cooperative advertising support to its retailer over a greater range of parameter values, than if that retailer were a monopolist. Interestingly, the manufacturer provides lower levels of advertising support as the intensity of competition increases, and in the limit, we obtain the monopoly result. This is because an increase in outside competition induces the supported retailer to advertise heavily of its own accord in order to wrest market share from its competing retailers, thus giving a free ride to the manufacturer.

Appendix B provides a brief discussion of two extensions. In the first, we consider the case where the manufacturer has the choice of selling through all of the retailers, and therefore has an option of supporting some or all of them. We show that when retailers are symmetric and one retailer provides a higher margin to the manufacturer than the others providing equal margins, then the manufacturer supports only the dominant retailer. It would be of interest to extend this analysis to asymmetric retailers. In the second extension, we incorporate retail price competition in the model. We did not consider the manufacturers wholesale price decision as in He et al. (2009) in the case of a single-manufacturer, single-retailer channel. Future research could examine the role of the wholesale price as another instrument for the manufacturer.

This paper also opens up a fruitful avenue for empirical research. Given the expression for the optimal subsidy rate as a function of various firm- and industry-level parameters, one could empirically examine whether our results can explain the differences in the subsidy rates in different industries as reported in Dutta et al. (1995). This would require estimation of the firm- and industry-level parameters, possibly employing the techniques used in Naik et al. (2008). Provided an appropriate data set can be found or collected, an empirical study to validate our results would undoubtedly deepen our understanding of the cooperative advertising practices in different industries.

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APPENDIX A

Proof of Proposition 1

The Hamilton-Jacobi-Bellman (HJB) equation for Retailer i is given by

$$rV_1 = \max_{u_1} \left\{ m_1 x - (1-\theta) u_1^2 + V_1^{'} \left(\rho_1 u_1 \sqrt{1-x} - \rho_2 u_2 \sqrt{x} - \delta_1 x + \delta_2 (1-x) \right) \right\}, \quad \text{(A1)}$$

$$rV_2 = \max_{u_2} \left\{ m_2 (1-x) - (1-\theta_2) u_2^2 + V_2^{'} \left(\rho_1 u_1 \sqrt{1-x} - \rho_2 u_2 \sqrt{x} - \delta_1 x + \delta_2 (1-x) \right) \right\}. \quad \text{(A2)}$$

The first-order conditions for maximization yield the optimal advertising levels in Equation (6) in Proposition 1(a). Substituting these solutions into the above HJB equations for the two retailers yields the following two equations:

$$rV_1 = m_1 x + \frac{\rho_1^2 \left(V_1^{'}\right)^2}{4\left(1 - \left(\frac{2V' - V_1^{'}}{2V' + V_1^{'}}\right)^{+}\right)} (1-x) + \frac{V_1^{'} V_2^{'} \rho_2^2}{2} x - V_1^{'} \left(\delta_1 x - \delta_2 (1-x)\right), \quad \text{(A3)}$$

$$rV_2 = m_2 (1-x) + \frac{\rho_2^2 \left(V_2^{'}\right)^2}{4} x + \frac{V_1^{'} V_2^{'} \rho_1^2}{2\left(1 - \left(\frac{2V' - V_1^{'}}{2V' + V_1^{'}}\right)^{+}\right)} (1-x) - V_2^{'} \left(\delta_1 x - \delta_2 (1-x)\right), \quad \text{(A4)}$$

The HJB equation for the manufacturer is given by

$$rV = \max_{\theta} \left\{ Mx - \theta u_1^2 (x|\theta) + V^{'} \left(\rho_1 u_1 (x|\theta) \sqrt{1-x} - \rho_2 u_2 (x|\theta) \sqrt{x} - \delta_1 x + \delta_2 (1-x) \right) \right\}. \quad \text{(A5)}$$

Substituting the optimal advertising efforts of the two retailers into the above equation and simplifying yields

$$rV = \max_{\theta} \left\{ Mx - \frac{\theta \rho_1^2 \left(V_1^{'}\right)^2 (1-x)}{4(1-\theta)^2} + V^{'} \left(\frac{\rho_1^2 V_1^{'} (1-x)}{2(1-\theta)} + \frac{\rho_2^2 V_2^{'} x}{2} - \delta_1 x + \delta_2 (1-x) \right) \right\}. \quad \text{(A6)}$$

Solving the first-order conditions for the subsidy rates yields Equation (7) in Proposition 1(b).

Substituting these solutions into Equation (A6) and simplifying, we obtain the following equation for the manufacturer:

$$rV = Mx - \frac{\rho_1^2 \left(V_1^{'}\right)^2 \left(\frac{2V^{'}-V_1^{'}}{2V^{'}+V_1^{'}}\right)^{+} (1-x)}{4\left(\left(1-\frac{2V^{'}-V_1^{'}}{2V^{'}+V_1^{'}}\right)^{+}\right)^2} + \frac{V^{'}V_2^{'}\rho_2^2 x}{2} - \frac{V^{'}V_1^{'}\rho_1^2(1-x)}{2\left(\left(1-\frac{2V^{'}-V_1^{'}}{2V^{'}+V_1^{'}}\right)^{+}\right)} - \delta_1 V^{'} x + \delta_2 V^{'}(1-x). \quad \blacksquare \tag{A7}$$

Proof of Proposition 2

With $V_1 = \alpha_1 + \beta_1 x$ and $V_2 = \alpha_2 + \beta_2 (1-x)$, we have $V_1^{'} = \beta_1$ and $V_2^{'} = -\beta_2$. Inserting these into Equations (A3) and (A4), we have

$$r(\alpha_1 + \beta_1 x) = m_1 x + \frac{\beta_1^2 \rho_1^2 (1-x)}{4\left(1-\left(\frac{2V^{'}-\beta_1}{2V^{'}+\beta_1}\right)^{+}\right)} - \frac{\beta_1 \beta_2 \rho_2^2 x}{2} - \beta_1 (\delta_1 x - \delta_2 (1-x)), \tag{A8}$$

$$r(\alpha_2 + \beta_2 (1-x)) = m_2 (1-x) + \frac{\beta_2^2 \rho_2^2 x}{4} - \frac{\beta_1 \beta_2 \rho_1^2 (1-x)}{2\left(1-\left(\frac{2V^{'}-\beta_1}{2V^{'}+\beta_1}\right)^{+}\right)} + \beta_2 (\delta_1 x - \delta_2 (1-x)). \tag{A9}$$

With $V = \alpha_M + \beta_M x$, we have $V^{'} = \beta_M$. Substitution into Equations (6) and (7) yields Equation (8) in Proposition 2(a) and Equation (9) in Proposition 2(b), respectively. Substituting these into the HJB equation (A7) and simplifying, we have

$$\begin{aligned} r(\alpha_M + \beta_M x) &= Mx - \frac{\beta_1^2 \rho_1^2 \left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^{+} (1-x)}{4\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^{+}\right)^2} - \frac{\beta_2 \beta_M \rho_2^2 x}{2} \\ &\quad + \frac{\beta_1 \beta_M \rho_1^2 (1-x)}{2\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^{+}\right)} - \delta_1 \beta_M x + \delta_2 \beta_M (1-x). \end{aligned} \tag{A10}$$

The system of equations to solve for the six value-function coefficients in Proposition 2(c) are

$$r\alpha_1 = \frac{\beta_1^2\rho_1^2}{4\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} + \beta_1\delta_2, \tag{A11}$$

$$r\beta_1 = m_1 - \frac{\beta_1^2\rho_1^2}{4\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} - \frac{\beta_1\beta_2\rho_2^2}{2} - \beta_1\left(\delta_1+\delta_2\right), \tag{A12}$$

$$r\alpha_2 = \frac{\beta_2^2\rho_2^2}{4} + \beta_2\delta_1, \tag{A13}$$

$$r\beta_2 = m_2 - \frac{\beta_2^2\rho_2^2}{4} - \frac{\beta_1\beta_2\rho_1^2}{2\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} - \beta_2\left(\delta_1+\delta_2\right), \tag{A14}$$

$$r\alpha_M = -\frac{\beta_1^2\rho_1^2\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+}{4\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)^2} + \frac{\beta_1\beta_M\rho_1^2}{2\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} + \delta_2\beta_M, \tag{A15}$$

$$r\beta_M = M + \frac{\beta_1^2\rho_1^2\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+}{4\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)^2} - \frac{\beta_2\beta_M\rho_2^2}{2} - \frac{\beta_1\beta_M\rho_1^2}{2\left(1-\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+\right)} - (\delta_1+\delta_2)\beta_M. \tag{A16}$$

We obtain these by substituting $V' = \beta_M$ into Equations (A8)-(A9) and equating the powers of x and $(1-x)$ in Equations (A8)-(A10). ■

Proof of Proposition 3

Setting $\left(\frac{2\beta_M-\beta_1}{2\beta_M+\beta_1}\right)^+ = 0$, we have:

$$
\begin{aligned}
r\alpha_1 &= \frac{\beta_1^2\rho_1^2}{4}+\beta_1\delta_2, && \text{(A17)}\\
r\beta_1 &= m_1-\frac{\beta_1^2\rho_1^2}{4}-\frac{\beta_1\beta_2\rho_2^2}{2}-\beta_1\left(\delta_1+\delta_2\right), && \text{(A18)}\\
r\alpha_2 &= \frac{\beta_2^2\rho_2^2}{4}+\beta_2\delta_1, && \text{(A19)}\\
r\beta_2 &= m_2-\frac{\beta_2^2\rho_2^2}{4}-\frac{\beta_1\beta_2\rho_1^2}{2}-\beta_2\left(\delta_1+\delta_2\right), && \text{(A20)}\\
r\alpha_M &= \frac{1}{2}\beta_1\beta_M\rho_1^2+\beta_M\delta_2, && \text{(A21)}\\
r\beta_M &= M-\frac{1}{2}\beta_1\beta_M\rho_1^2-\frac{1}{2}\beta_2\beta_M\rho_2^2-\beta_M\left(\delta_1+\delta_2\right). && \text{(A22)}
\end{aligned}
$$

Solving Equation (A18) for β_2 yields

$$
\beta_2 = \frac{4m_1-\beta_1\left(\beta_1\rho_1^2+4\left(r+\delta_1+\delta_2\right)\right)}{2\beta_1\rho_2^2}. \tag{A23}
$$

Substituting the above solution into Equation (A20) and simplifying yields the quartic Equation (11). As detailed in Prasad and Sethi (2004), it can be shown that there exists a unique $\beta_1 > 0$ that solves the above quartic equation. This is the unique equilibrium of the Stackelberg differential game. That solution can then be substituted into Equation (A23) to obtain the solution for β_2. We know from Equation (A22) that

$$
\beta_M = \frac{2M}{\beta_1\rho_1^2+\beta_2\rho_2^2+2\left(r+\delta_1+\delta_2\right)}. \tag{A24}
$$

Substituting the solutions for β_1 and β_2 into Equation (A24) yields β_M. Since $\beta_M > 0$ from Equation (A16), the subsidy threshold is obtained as $2\beta_M-\beta_1$. We have

$$
\frac{4M}{\beta_1\rho_1^2+\beta_2\rho_2^2+2\left(r+\delta_1+\delta_2\right)}-\beta_1 = \frac{4M-\beta_1^2\rho_1^2-\beta_1\beta_2\rho_2^2-2\beta_1\left(r+\delta_1+\delta_2\right)}{\beta_1\rho_1^2+\beta_2\rho_2^2+2\left(r+\delta_1+\delta_2\right)}. \tag{A25}
$$

We know from Equation (A18) that

$$\beta_1\beta_2\rho_2^2 + 2\beta_1(\delta_1+\delta_2) = \frac{1}{2}\left(4m_1 - 4r\beta_1 - \beta_1^2\rho_1^2\right).$$

Using this in the numerator of Equation (A25) and simplifying gives the subsidy threshold S in Equation (10). Since β_1 is independent of M, we have $\partial S/\partial M = 2 > 0$. ■

Proof of Corollary 1

The solutions for the value-function coefficients can be found in He et al. (2011). For the comparative statics, imposing symmetry in the subsidy threshold from Equation (10), we have $S(\cdot) = 2M - m - \frac{\beta^2\rho^2}{4} = 0$. In this threshold, we substitute the solution for β, as specified in He et al. (2011) to obtain

$$S(\cdot) = 2M - \frac{4}{3}m + \frac{2(r+2\delta)\left(\sqrt{(r+2\delta)^2+3m\rho^2} - (r+2\delta)\right)}{9\rho^2}. \tag{A26}$$

Taking the derivative of S in Equation (A26) w.r.t. the model parameters yields the comparative statics in Corollary 1(a): $\partial S/\partial M = 2 > 0$, $\partial S/\partial m = \frac{r+2\delta}{3\sqrt{(r+2\delta)^2+3m\rho^2}} - \frac{4}{3} < 0$, $\partial S/\partial\rho = \frac{2m(r+2\delta)}{3\rho\sqrt{(r+2\delta)^2+3m\rho^2}} - \frac{4(r+2\delta)\left(\sqrt{(r+2\delta)^2+3m\rho^2}-r-2\delta\right)}{9\rho^3} < 0$, $\partial S/\partial r = \frac{2\left(\sqrt{(r+2\delta)^2+3m\rho^2}-r-2\delta\right)^2}{9\rho^2\sqrt{(r+2\delta)^2+3m\rho^2}} > 0$, and $\partial S/\partial\delta = \frac{4\left(\sqrt{(r+2\delta)^2+3m\rho^2}-r-2\delta\right)^2}{9\rho^2\sqrt{(r+2\delta)^2+3m\rho^2}} > 0$.

For any two parameters ψ and χ, the implicit function theorem yields $\text{sign}[\partial\psi/\partial\chi] = \text{sign}[-\frac{\partial S/\partial\chi}{\partial S/\partial\psi}]$. Thus, $-\frac{\partial S/\partial m}{\partial S/\partial(M)} = \frac{1}{2}\left(\frac{4}{3} - \frac{r+2\delta}{3\sqrt{(r+2\delta)^2+3m\rho^2}}\right) > 0$, $-\frac{\partial S/\partial m}{\partial S/\partial\rho} = -\frac{3\rho^3\left(4\sqrt{(r+2\delta)^2+3m\rho^2}\right)-(r+2\delta)}{2(r+2\delta)\left(3m\rho^2-2(r+2\delta)\left(\sqrt{(r+2\delta)^2+3m\rho^2}\right)-(r+2\delta)\right)} < 0$, $-\frac{\partial S/\partial\delta}{\partial S/\partial M} = -\frac{2\left(\sqrt{(r+2\delta)^2+3m\rho^2}-(r+2\delta)\right)^2}{9\rho^2\sqrt{(r+2\delta)^2+3m\rho^2}} < 0$, $-\frac{\partial S/\partial\delta}{\partial S/\partial m} = \frac{4\left(\sqrt{(r+2\delta)^2+3m\rho^2}-(r+2\delta)\right)^2}{3\rho^2\left(4\sqrt{(r+2\delta)^2+3m\rho^2}-(r+2\delta)\right)} > 0$, $-\frac{\partial S/\partial\delta}{\partial S/\partial\rho} = \frac{2\rho}{r+2\delta} > 0$, and $-\frac{\partial S/\partial\delta}{\partial S/\partial r} = -2 < 0$. The comparative statics results in Corollary 1(b) follow. ■

Proof of Proposition 4

The HJB equation for Retailer 1 is given by

$$rV_1 = \max_{u_1} \begin{cases} m\left(1-\sum_{j=2}^{n} x_j\right) - (1-\theta)u_1^2 \\ \quad +\sum_{j=2}^{n} \frac{\partial V_1}{\partial x_j}\left(\frac{n\rho}{n-1}u_j\sqrt{1-x_j} - \frac{1}{n-1}\sum_{k=1}^{n}\rho u_k\sqrt{1-x_k} - \delta\left(x_j-\frac{1}{n}\right)\right). \end{cases} \tag{A27}$$

The HJB equation for Retailer i, $i \in \{2,3,...,n\}$, is given by

$$rV_i = \max_{u_1} \left\{ mx_i - u_i^2 + \sum_{j=2}^{n} \frac{\partial V_i}{\partial x_j}\left(\frac{n\rho}{n-1}u_j\sqrt{1-x_j} - \frac{1}{n-1}\sum_{k=1}^{n}\rho u_k\sqrt{1-x_k} - \delta\left(x_j-\frac{1}{n}\right)\right). \right. \tag{A28}$$

The first-order conditions for maximization yield the optimal advertising levels in Equations (22)-(23) in Proposition 4(a). The HJB equation for the manufacturer is given by

$$rV = \max_{\theta(t)} \begin{cases} M(1-\sum_{j=2}^{n} x_j) - \theta u_1^2 \\ \quad +\sum_{j=2}^{n} \frac{\partial V}{\partial x_j}\left(\frac{n}{n-1}\rho u_j\sqrt{1-x_j} - \frac{1}{n-1}\sum_{k=2}^{n}\rho u_k\sqrt{1-x_k} - \delta\left(x_j-\frac{1}{n}\right)\right). \end{cases} \tag{A29}$$

Substituting the optimal advertising efforts of the n retailers, specified in Equations (22)-(23), into Equation (A29) and simplifying yields

$$rV = \max_{\theta(t)} \begin{cases} M(1-\sum_{j=2}^{n} x_j) - \frac{\theta\rho^2\sum_{j=2}^{n} x_j}{4(n-1)^2(1-\theta)^2}\left(\sum_{k=2}^{n}\frac{\partial V_1}{\partial x_k}\right)^2 \\ \quad +\frac{\rho^2\sum_{j=2}^{n} x_j}{2(n-1)^2(1-\theta)}\left(\sum_{k=2}^{n}\frac{\partial V_1}{\partial x_k}\right)\left(\sum_{j=2}^{n}\frac{\partial V}{\partial x_j}\right) \\ \quad +\sum_{j=2}^{n}\frac{\rho^2(1-x_j)}{2(n-1)^2}\left(n\frac{\partial V}{\partial x_j} - \sum_{k=2}^{n}\frac{\partial V}{\partial x_k}\right)\left(n\frac{\partial V_j}{\partial x_j} - \sum_{k=2}^{n}\frac{\partial V_j}{\partial x_k}\right) - \sum_{j=2}^{n}\frac{\partial V}{\partial x_j}\delta\left(x_j-\frac{1}{n}\right). \end{cases} \tag{A30}$$

Using the first-order condition of Equation (A30) w.r.t. θ and simplifying yields Proposition 4(b). Substituting Equations (22)-(24) into Equations (A27)-(A28) and (A30) yields the following simultaneous equations satisfied by the value functions V_i for Retailer i, $i = 1,2,...,n$, and V for the manufacturer:

$$rV_1 = m\left(1-\sum_{j=2}^{n} x_j\right) + \frac{\rho^2 \sum_{j=2}^{n} x_j}{4(n-1)^2\left(1-\left(\frac{2B-A}{2B+A}\right)^+\right)}\left(\sum_{k=2}^{n}\frac{\partial V_1}{\partial x_k}\right)^2$$
$$+\sum_{j=2}^{n}\frac{\rho^2(1-x_j)}{2(n-1)^2}\left(n\frac{\partial V_1}{\partial x_j}-\sum_{k=2}^{n}\frac{\partial V_1}{\partial x_k}\right)\left(n\frac{\partial V_j}{\partial x_j}-\sum_{k=2}^{n}\frac{\partial V_j}{\partial x_k}\right)-\sum_{j=2}^{n}\frac{\partial V_1}{\partial x_j}\delta\left(x_j-\frac{1}{n}\right). \quad \text{(A31)}$$

$$\begin{aligned} rV_i &= mx_i + \frac{\rho^2(1-x_i)}{4(n-1)^2}\left(n\frac{\partial V_i}{\partial x_i}-\sum_{k=2}^{n}\frac{\partial V_i}{\partial x_k}\right)^2 + \frac{\rho^2\sum_{j=2}^{n}x_j}{2(n-1)^2\left(1-\left(\frac{2B-A}{2B+A}\right)^+\right)}\sum_{j=2}^{n}\frac{\partial V_i}{\partial x_j}\left(\sum_{k=2}^{n}\frac{\partial V_1}{\partial x_k}\right) \\ &+ \sum_{j\neq i, j=2}^{n}\frac{\rho^2(1-x_j)}{2(n-1)^2}\left(n\frac{\partial V_i}{\partial x_j}-\sum_{k=2}^{n}\frac{\partial V_i}{\partial x_k}\right)\left(n\frac{\partial V_j}{\partial x_j}-\sum_{k=2}^{n}\frac{\partial V_j}{\partial x_k}\right) \\ &- \sum_{j=2}^{n}\frac{\partial V_i}{\partial x_j}\delta\left(x_j-\frac{1}{n}\right), i=2,3,...,n. \end{aligned} \quad \text{(A32)}$$

$$\begin{aligned} rV &= Mx_1 - \frac{\rho^2\sum_{j=2}^{n}x_j}{4(n-1)^2\left(1-\left(\frac{2B-A}{2B+A}\right)^+\right)^2}\left(\frac{2B-A}{2B+A}\right)^+\left(\sum_{k=2}^{n}\frac{\partial V_1}{\partial x_k}\right)^2 \\ &+ \frac{\rho^2\sum_{j=2}^{n}x_j}{2(n-1)^2\left(1-\left(\frac{2B-A}{2B+A}\right)^+\right)}\left(\sum_{k=2}^{n}\frac{\partial V_1}{\partial x_k}\right)\left(\sum_{j=2}^{n}\frac{\partial V}{\partial x_j}\right) \\ &+ \sum_{j=2}^{n}\frac{\rho^2(1-x_j)}{2(n-1)^2}\left(n\frac{\partial V}{\partial x_j}-\sum_{k=2}^{n}\frac{\partial V}{\partial x_k}\right)\left(n\frac{\partial V_j}{\partial x_j}-\sum_{k=2}^{n}\frac{\partial V_j}{\partial x_k}\right)-\sum_{j=2}^{n}\frac{\partial V}{\partial x_j}\delta\left(x_j-\frac{1}{n}\right). \end{aligned} \quad \text{(A33)}$$

In total, Equations (A31)-(A33) are $(n+1)$ simultaneous equations satisfied by the value functions V_i, $i=1,2,...,n$, and V. ■

Proof of Proposition 5

With $V = P - Q\sum_{i=2}^{n} x_i$, $V_1 = a_1 - b_1\sum_{i=2}^{n} x_i$, $V_i = a - b + bx_i$, for $i = 2,3,...,n$, we have

$$\frac{\partial V}{\partial x_i} = -Q, \quad \frac{\partial V_1}{\partial x_i} = -b_1, \quad \frac{\partial V_i}{\partial x_i} = b, \quad \frac{\partial V_i}{\partial x_j} = 0, \quad i,j = 2,3,...,n, \quad i \neq j. \quad \text{(A34)}$$

Inserting Equation (A34) into Equations (22)-(23), we obtain Equation (26) in Proposition 5(a). Inserting Equation (A34) into Equation (24), we obtain Equation (27) in Proposition 5(b). Substituting the linear value functions into Equations (A31)-(A32), we have

$$\begin{aligned} r\left(a_1 - b_1\sum_{j=2}^{n} x_j\right) &= m\left(1-\sum_{j=2}^{n} x_j\right) + \frac{b_1^2\rho^2\sum_{j=2}^{n} x_j}{4\left(1-\left(\frac{2Q-b_1}{2Q+b_1}\right)^+\right)} \\ &\quad - \sum_{j=2}^{n}\frac{bb_1\rho^2(1-x_j)}{2(n-1)} + b_1\delta\sum_{j=2}^{n}\left(x_j-\frac{1}{n}\right), \end{aligned} \tag{A35}$$

$$\begin{aligned} r(a-b+bx_i) &= mx_i + \frac{b^2\rho^2(1-x_i)}{4} - \frac{bb_1\rho^2\sum_{j=2}^{n} x_j}{2(n-1)\left(1-\left(\frac{2Q-b_1}{2Q+b_1}\right)^+\right)} \\ &\quad - \sum_{j\neq i, j=2}^{n}\frac{b^2\rho^2(1-x_j)}{2(n-1)} - \delta bx_i + \frac{\delta b}{n}. \end{aligned} \tag{A36}$$

Equating the coefficient of the term $(\sum_{j=2}^{n} x_j)$ in Equation (A35) and the coefficient of x_i in Equation (A36), we obtain the equations

$$ra_1 = m - \frac{(n-1)b_1\delta}{n)} - \frac{b_1 b\rho^2}{2}, \tag{A37}$$

$$(r+\delta)b_1 = m - \frac{b_1^2\rho^2}{4\left(1-\left(\frac{2Q-b_1}{2Q+b_1}\right)^+\right)} - \frac{b_1 b\rho^2}{2(n-1)}, \tag{A38}$$

$$r(a-b) = \frac{b^2\rho^2}{4} - \frac{(n-2)b^2\rho^2}{2(n-1)} + \frac{\delta b}{n}, \tag{A39}$$

$$(r+\delta)b = m - \frac{b^2\rho^2}{4} - \frac{b_1 b\rho^2}{2(n-1)\left(1-\left(\frac{2Q-b_1}{2Q+b_1}\right)^+\right)}. \tag{A40}$$

Inserting the linear value functions into Equation (A33), we have

$$\begin{aligned} r\left(P - Q\sum_{j=2}^{n} x_j\right) &= M\left(1-\sum_{j=2}^{n} x_j\right) - \frac{b_1^2\rho^2\left(\sum_{j=2}^{n} x_j\right)\left(\frac{2Q-b_1}{2Q+b_1}\right)^+}{4(\left(1-\left(\frac{2Q-b_1}{2Q+b_1}\right)^+\right)} + \frac{Qb_1\rho^2\left(\sum_{j=2}^{n} x_j\right)}{2\left(1-\left(\frac{2Q-b_1}{2Q+b_1}\right)^+\right)} \\ &\quad - \sum_{j=2}^{n}\frac{Qb\rho^2(1-x_j)}{2(n-1)} + \sum_{j=2}^{n} Q\delta\left(x_j-\frac{1}{n}\right). \end{aligned} \tag{A41}$$

Equating the coefficients of the term $(\sum_{j=2}^{n} x_j)$ in Equation (A41), we obtain

$$rP = M - \frac{(n-1)Q\delta}{n} - \frac{Qb\rho^2}{2}, \tag{A42}$$

$$(r+\delta)Q = M + \frac{b_1^2\rho^2\left(\frac{2Q-b_1}{2Q+b_1}\right)^+}{4\left(1-\left(\frac{2Q-b_1}{2Q+b_1}\right)^+\right)^2} - \frac{Qb_1\rho^2}{2\left(1-\left(\frac{2Q-b_1}{2Q+b_1}\right)^+\right)} - \frac{Qb\rho^2}{2(n-1)}. \quad \blacksquare \tag{A43}$$

Proof of Corollary 2

Setting $\theta^* = 0$, $a_1 = a$, and $b_1 = b$ in Equations (A40)-(A43) and solving for a, b, P, and Q, we obtain Equation (29) and Equation (31) in Corollary 2. ■

Proof of Proposition 6

Since $b > 0$ and $Q > 0$, the sign of $\left(\frac{2Q-b}{2Q+b}\right)$ is the same as the sign of $2Q/b - 1$. Therefore, we set the threshold $S = 2Q/b - 1$. ■

Proof of Corollary 3

This result can be obtained by setting $r = \delta = 0$ in Proposition 6. ■

APPENDIX B

Selling Through All Retailers

Up to this point, we have assumed that only one retailer is supported by the manufacturer. This might be the case if a manufacturer sells to retailers whose locations are dispersed. Suppose now that the manufacturer sells through all retailers. A noteworthy case arises when the manufacturer's margin is equal across all n retailers. Obviously, if this common margin is M, then the manufacturer captures the whole market at this margin regardless of the individual market shares of the retailers

and their parameters. It is then clear that the manufacturer has no need to support any of the retailers $\theta_i^* = 0$, $i = 1,2,...,n$. Then, clearly the value function of the manufacturer is

$$\int_0^\infty e^{-rt}(Mx(t) + M(1 - x(t)))dt = \int_0^\infty e^{-rt} M dt = M/r.$$

It is possible that there are other parameter combinations for which $\theta_i^* = 0$, $i = 1,2,...,n$, which need to be derived to allow us to understand those situations where a manufacturer does not provide any advertising support to its retailers. For the cases where $\theta_i^* = 0$, $i = 1,2,...,n$, we can see that the manufacturer's and the retailers' problems are decoupled. Having solved the manufacturer's problem, we see that the retailers' problem is reduced to the Nash differential game that was solved in Prasad and Sethi (2004).

Next, we take up the case when the manufacturer's margins are not equal across all retailers. Here, we limit our analysis to the situation where retailers are symmetric, as before, but $M_i > 0$, $i = 2,3,...,n$. Instead we assume that the manufacturer sells to Retailer 1 at a margin M and to the other symmetric retailers at a common margin $M_i = \mu > 0$, $i \in \{2,3,...,n\}$. The manufacturer sets the participation rates for Retailer i at $\theta_i(x_2, x_3, ..., x_n)$ and then retailers choose advertising efforts u_i. Thus, Retailer i maximizes

$$V_i = \max_{u_i(t),\, t \geq 0} \int_0^\infty e^{-rt} \left(m x_i - (1 - \theta_i(x_2(t), x_3(t), ..., x_n(t)))\, u_i^2(t) \right) dt, \quad i = 1,2,...n. \tag{A44}$$

Solving the Nash differential game with the objective functions in Equation (A44) yields Retailer i's feedback advertising effort, denoted $u_i(x_2, x_3, ..., x_n | \theta_1, \theta_2, ..., \theta_n)$. The manufacturer anticipates the retailers' reaction functions when solving for its subsidy rates. Therefore, the manufacturer's problem is given by

$$V = \max_{\theta_1(t),\, \theta_2(t), ...,\, \theta_n(t)\, t \geq 0} \int_0^\infty e^{-rt} \left(Mx(t) + \mu(1 - x(t)) - \sum_{i=1}^n \theta_i(t) u_i^2(t) \right) dt \tag{A45}$$

subject to the retail dynamics.

For the Stackelberg-Nash game defined by Equations (A44), (A45), and (18), we have the following result.

Proposition 7: For symmetric retailers, it is never optimal for the manufacturer to support all retailers. In particular, when $M > \mu \geq 0$, only the first retailer may be supported $(\theta_1^* \geq 0)$ and $\theta_i^* = 0$, $i \in \{2,3,...,n\}$.

Proof of Proposition 7

First, note that if $M > \mu$, the manufacturer's objective function in equation (A45) can be organized as follows:

$$\int_0^\infty e^{-rt}\left(\mu + (M-\mu)x - \theta_1 u_1^2(x) - \sum_{j=2}^{n} \theta_j u_j^2(x)\right) dt.$$

This means that the manufacturer should encourage Retailer 1 to increase advertising in order to increase market share (and discourage the other retailers from advertising by setting $\theta_j^* = 0$, $j \in \{2,3,...,n\}$). ■

To summarize, we have treated two special cases of selling through all retailers in this section. If manufacturer's margins are equal across retailers, no one gets a subsidy. If the manufacturer has a higher margin from Retailer 1 than it has from all other retailers, which are assumed to be symmetric and to be providing equal margins to the manufacturer, then the only retailer that might be supported is Retailer 1. The case with asymmetric retailers is rather intractable, although some other special cases may be solvable.

Retail Price Competition

We present an extension of the model specified by Equations (1)-(5) that includes retail (but not wholesale) price competition to show the robustness of the results to the inclusion of price. We let

the retailers choose their prices after deciding their advertising levels. The optimal prices resulting from this formulation are then substituted to yield the retailers margins. This is similar to the Bass et al. (2005) approach. Thus, the model with endogenous retail price competition reduces to the current model with exogenous margins.

Retailer i's discounted profit maximization problem, $i = 1,2$, is given by

$$V_i(x_0) = \max_{u_i(t),\, p_i(t)} \int_0^\infty e^{-rt} \left(D_i(p_i(t), p_{3-i}(t)) (p_i(t) - c_i) x_i(t) - (1 - \theta_i(x(t))) u_i^2(t) \right) dt, \quad \text{(A46)}$$

subject to Equation (1), where $p_i(t)$ is the price charged by Retailer i,

$$D_i(p_i(t), p_{3-i}(t)) = (1 - h_i p_i(t) + d_{3-i} p_{3-i}(t))$$

is the demand function expressed as a function of the prices of the two retailers, c_i is the marginal cost of Retailer i, and h_i and d_i are demand parameters. Solving for the retailers' optimal prices yields

$$\hat{p}_i = \frac{d_{3-i} + h_{3-i}(2 + 2h_i c_i + d_{3-i} c_{3-i})}{4h_1 h_2 - d_1 d_2}, \quad i = 1,2. \quad \text{(A47)}$$

Using the parameter m_i to denote $(D_i(p_i(t), p_{3-i}(t)))(p_i(t) - c_i)$ in Equation (A46), where $p_i = \hat{p}_i$ from Equation (A47), results in Equations (2)-(3). The demand parameters affect price, which in turn affects the profit margins. Thus, the results in the Analysis and Results section remain unchanged except that we can now add that the demand parameters affect the subsidy rate θ^* via m_i, $i \in \{1,2\}$.

Table A1: Notation.

t	Time t, $t \geq 0$
$x(t) \in [0,1]$	Retailer 1's share of the market at time t
x_0	Initial market share of Retailer 1
$u_i(t) \geq 0$	Retailer i's advertising effort rate at time t, $i = 1,2$
$\theta(t) \geq 0$	Manufacturer's subsidy rate for Retailer 1 at time t
$\rho_i > 0$	Advertising effectiveness parameter
$\delta_i \geq 0$	Market share decay parameter
$r > 0$	Discount rate of the retailers and the manufacturer
$m_i > 0$	Gross margin of Retailer i
$M \geq 0$	Gross margin of the manufacturer from Retailer 1
V_i, V	Value functions of Retailer i and the manufacturer, respectively
$f'(x)$ or f'	df/dx for a differentiable function $f(x)$
$f(x)^+$ or f^+	$\max\{f(x),0\}$ for a function $f(x)$

Figure 1: Market structure.

Figure 2: Effect of retailers' margins on the optimal subsidy rate.

Figure 3: Effect of advertising effectiveness on the optimal subsidy rate.

Figure 4: Effect of decay rate on the optimal subsidy rate.

Figure 5: Effect of competition on the optimal subsidy rate.