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SHORT COMMUNICATIONS

OLIGOPOLY ADVERTISING STRATEGIES WITH MARKET EXPANSION

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SUMMARY

This study extends the literature of dynamic advertising competition by allowing oligopoly competition and market expansion that results from advertising. We find how advertising actions affect market development and how, conversely, market development influences advertising policies over time. Modelling the competition by a differential game, we solve for both time-variant closed-loop and time-invariant feedback Nash equilibrium strategies. The time-variant closed-loop strategy depends on a firm's own sales rates and the time-invariant feedback strategy depends on the growth of the market. We discuss the marketing implications of the results. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: Advertising-competitive strategy; Nash equilibrium; differential games; market expansion; non-co-operative

1. INTRODUCTION

This study attempts to determine the optimal (profit-maximizing) levels of investments in marketing activities (such as advertising) in a competitive and dynamic market which has the potential for market growth. Empirical examples illustrating the importance of advertising in competitive markets can be found in Carpenter *et al.*[1] and Little.[2] We consider the case where the players focus their marketing activities to attract sales from each other and from outside the competition.

Differential games provide an appropriate framework for analysing dynamic marketing expenditure decisions in a competitive setting. We chose the Lanchester combat model to describe the dynamics of the competition resulting from firms' marketing efforts to attract sales from competitors. This model was first applied by Kimball[3] to a combat problem, then by Vidale and Wolfe[4] as a sales-advertising response model, and by Isaacs,[5] Horsky,[6] Little,[2], Case,[7] Deal *et al.*,[8] Deal,[9] Sorger,[10] Chintagunta and Vilcassim,[11] Erickson,[12–15] and Fruchter and Kalish[16] to find optimal advertising strategies in a dynamic, competitive market. Chintagunta and Vilcassim[11] and Erickson[14] provide empirical evidence that a closed-loop solution fits the data better than an open-loop solution.

A drawback of all the above studies is that they make the key assumption of fixed total industry sales. This assumption limits the model to a mature, saturated market, i.e. where the total

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saturated industry sales are the same as the total current industry sales. Moreover, this strong assumption ignores the possibility that advertising by itself can expand the whole target market. The present paper sheds light on this issue by addressing questions such as: Does competitive advertising extend the market? If the answer is yes; by how much? How much should an oligopolist invest in advertising under such conditions?

Developing a dynamic sales response model for oligopolistic competition, we show that the size of the whole market grows exponentially with the aggregate effectiveness of *all* competitors' efforts. Generating a suitable differential game we derive open- and closed-loop Nash equilibrium strategies, and explain the relationship between them. The closed-loop strategy of each player has the property of being proportional to the open-loop strategy, which is time-variant, and to the square of the player's potential target market. This structure provides a formula for a practical adaptive control rule. Finally, we find a time-invariant Nash equilibrium feedback strategy that depends on the growth of the market, which has an analytic solution in the case of zero discount. We summarize the contributions of this paper as follows: (a) it generalizes the previous duopoly results of Fruchter and Kalish[16] to an oligopoly which allows market expansion; (b) it provides a time-invariant feedback Nash equilibrium solution for oligopoly growing markets.

2. MODEL DESCRIPTION

We consider an industry in an oligopoly market where n competitors use advertising (as their major marketing instrument) to attract customers from each other as well as from the new customers, in their efforts to maximize their discounted profits over the planning horizon. Assuming an infinite planning horizon, the problem of oligopolist k, $k \in \{1, \ldots, n\}$, is to develop an advertising strategy that will maximize the present value of the firm's profit stream, as given by

$$\Pi_k = \int_0^\infty (q_k s_k(t) - u_k^2(t)) \mathrm{e}^{-rt}\,\mathrm{d}t, \quad k = 1, \ldots, n \tag{1}$$

In formula (1), $s_k(t)$ represents the sales rate of firm k at time t; q_k is the gross profit rate; the quantity $u_k^2(t)$ represents the advertising expenditure of firm k at time t, identifying u_k (the square root of the advertising expenditure of firm k) with the advertising effort of the competitor k; and r is the discount rate. For simplicity we assume that all the firms have the same discount rate.

There is a dynamic relation between the rate of change in sales and the competitors' simultaneous advertising efforts to attract and generate sales. In a duopoly, a well-known relation which captures the above dynamics in the form of a differential equation, is the Lanchester combat model. Previous studies have used this model only under the restriction of fixed market size. This assumption limits the model to the particular case of a mature market, where marketing activities have the potential only to switch customers from one competitor to another, i.e. where the competing firms deal only with market penetration.

Here, we dispense with this strong limitation, and broadening the Lanchester model† to oligopoly competition, we allow for the situation in which marketing activities have the potential to generate *new customers*, that is to bring about *market expansion*. Let the term $\rho_\kappa u_k(t)$ denote the effectiveness of the advertising efforts of firm k, $k \in \{1, 2, \ldots, n\}$, in attracting potential sales at time t. The constant ρ_κ is related to media of advertising and to consumers' brand perceptions and

† The literature adopting the Lanchester model usually works with market shares. Since here we work with sales, the extended model which will be introduced can perhaps be more related to the Vidale-Wolfe[4] model.

preferences. The model would be more accurate if we could differentiate between the effectiveness of advertising in attracting customers from each of the competitors and in generating new customers. However, if we allowed for this distinction, we would obtain a more complex mathematical problem. Moreover, the number of parameters in the model would increase with the number of brands competing in the market. Difficulty in estimating all the parameters would render a multi-parameter model unworkable, and so we sacrifice this refinement in favour of simplicity and empirical implementability.

Let m be a parameter which represents the *saturation level* of the market or, in other words, the limit of the entire industry sales that can be generated as a result of marketing activities. Let $\varepsilon(t)$ be the *market potential* at time t, i.e. the difference between the saturation level of the market and the total present industry sales at time t. Then

$$m = \sum_{k=1}^{n} s_k(t) + \varepsilon(t) \tag{2}$$

Using the above notation in order to examine changes in the total industry sales as a result of marketing activities, we consider the extended Lanchester oligopoly model

$$\dot{s}_k(t) = \rho_k u_k(t)(m - s_k(t)) - s_k(t) \sum_{\substack{j=1 \\ j \neq k}}^{n} \rho_j u_j(t), \quad s_k(0) = s_k^0, \quad k = 1, \dots, n \tag{3}$$

In other words, changes in the sale rates of brand k are a result of the advertising efforts on potential sales of firm k and the simultaneous opposite efforts of the rivals to generate brand switching. This model does not include customer retention activities.

2.1. Market expansion

Taking the derivative of (2) with respect to t and making use of (3), we obtain

$$\dot{\varepsilon}(t) = -\sum_{k=1}^{n} \dot{s}_k(t) = -\varepsilon(t) \sum_{k=1}^{n} \rho_k u_k(t), \quad \varepsilon(0) = m - \sum_{k=1}^{n} s_k^0 \tag{4}$$

Integrating (4), we obtain

$$\varepsilon(t) = \varepsilon(0) \mathrm{Exp}\left[-\sum_{k=1}^{n} \int_0^t \rho_k u_k(\tau)\, \mathrm{d}\tau \right] \tag{5}$$

where $\int_0^t \rho_k u_k(\tau)\, \mathrm{d}\tau$ represents the *cumulative advertising effectiveness of firm k's efforts.*

Remark 1

Since $\varepsilon(t) = m - \sum_{k=1}^{m} s_k(t)$ represents the market potential at time t, formula (5) *demonstrates* that the *market potential in our model is a function of advertising efforts of all players.*

Denote the total industry sales by $I(t)$, i.e.

$$I(t) = \sum_{k=1}^{n} s_k(t) = m - \varepsilon(t) \tag{6}$$

Considering equations (5) and (6) we obtain the following result.

Proposition 1

The total industry sales, $I(t)$, increases exponentially with the aggregate cumulative advertising effectiveness of all the competitors' efforts.

In other words, the competition and accompanying advertising activities are the *factors* that contribute to market expansion. We also deduce that when there are marketing activities with positive effects the market always increases up to the saturation level. In the case of negative effects the total industry may decline. Considering (5) and (6), we have

$$\varepsilon(\infty) = 0 \text{ and } I(\infty) = m \tag{7}$$

which means that in case of positive effects of advertising, the market will attain its saturation level at the steady state.

2.2. Fixed market size

The particular case of *fixed total industry sales* corresponds to

$$\dot{I}(t) = 0, \quad I(0) = \sum_{k=1}^{n} s_k^0. \tag{8}$$

Considering (4) and (6), this may happen if $I(t) = m$, i.e. when the market is at its saturated level. In other words, the assumption made in earlier Lanchester models, that the total industry is fixed, is a particular case of our new model. Since the condition of fixed market size is a strong assumption, the proposed model enables us to model conditions that occur commonly in the marketplace.

2.3. The differential game

Considering equations (1) and (3), we obtain that oligopolist k's problem, $k = 1, \ldots, n$, is

$$\begin{aligned} &\operatorname*{Max}_{u_k} \Pi_k = \int_0^\infty (q_k s_k(t) - u_k^2(t)) \mathrm{e}^{-rt} \, \mathrm{d}t \\ &\text{s.t.} \quad \dot{s}_k(t) = \rho_k u_k(t) m - s_k(t) \sum_{j=1}^{n} \rho_j u_j(t), \quad s_k(0) = s_k^0 \end{aligned} \tag{9}$$

Let

$$x_k = \frac{s_k}{m} \tag{10a}$$

be the fraction of the brand k sales relative to the saturation level of the market, or the *saturated market share* of firm k, and let

$$Q_k = q_k m \tag{10b}$$

then (9) becomes

$$\begin{aligned} &\operatorname*{Max}_{u_k} \Pi_k = \int_0^\infty (Q_k x_k(t) - u_k^2(t)) \mathrm{e}^{-rt} \, \mathrm{d}t \\ &\text{s.t.} \quad \dot{x}_k(t) = \rho_k u_k(t) - x_k(t) \sum_{j=1}^{n} \rho_j u_j(t), \quad x_k(0) = x_k^0 \end{aligned} \tag{11}$$

A control function $u_k(t)$, $k = 1, \dots, n$, is an admissible control if it is nonnegative and bounded in $[0, \infty)$. For such controls, the state equation reveals that $0 \leqslant s_k(t) \leqslant m$.

For the differential game (9) we want to find *Nash equilibrium closed-loop strategies*, that is strategies which are responses to market measurements, s_k, $k = 1, \dots, n$. Such strategies have the ability to capture current changes in the market, i.e. they are realistic strategies for a dynamic competitive market. In addition, by being responsive to actual measurements of sales, they have the advantage of capturing changing conditions in the market that are not captured by the model.

3. DETERMINATION OF THE CLOSED-LOOP STRATEGIES

We now present the derivation of closed-loop strategies, i.e., strategies of the form $u_k^* = u_k^*(t, s_1, \dots, s_n, s_1^0, \dots, s_n^0)$. For this objective we use formulation (11) for the differential game. Let $\varphi(t)$ satisfy the following two-point boundary-value problem (TPBVP):

$$\dot{x}_k^*(t) = \frac{1}{2}\left[\rho_k^2 Q_k \varphi(t)(1 - x_k^*(t)) - x_k^*(t) \sum_{j=1}^{n} \rho_j^2 Q_j \varphi(t)(1 - x_j^*(t))\right] \mathrm{e}^{rt}, \quad x_k^*(0) = x_k^0, \quad k = 1, \dots, n$$

$$\dot{\varphi}(t) = \frac{1}{2} \sum_{j=1}^{n} \rho_j^2 Q_j (1 - x_j^*(t)) \varphi^2(t) \mathrm{e}^{rt} - \mathrm{e}^{-rt}, \quad \varphi(\infty) = 0 \tag{12}$$

Let x_k be as in (11). Then, as shown in the Appendix I, we are able to construct the time-variant closed-loop strategies

$$u_k^* = \tfrac{1}{2} \rho_k Q_k \varphi(t) \mathrm{e}^{rt} (1 - x_k), \quad k = 1, \dots, n \tag{13}$$

or

$$u_k^* = \tfrac{1}{2} \rho_k q_k \varphi(t) \mathrm{e}^{rt} (m - s_k), \quad k = 1, \dots, n \tag{14}$$

where s_k is as in (9).

Next, we prove that the closed-loop strategy defined in (13) (or (14)) forms a global Nash equilibrium strategy for the differential game associated with (11) (or (9)).

Theorem 1

Consider the differential game associated with (9) *or* (11)). *Then* $(u_1^*, \dots, u_n^*)$ *defined in* (13) *and/or* (14) *forms a global Nash equilibrium closed-loop strategy for the above differential game.*

Proof. See Appendix II. □

3.1. *The open-loop vs. the closed-loop: discussion*

As shown in Appendix I, the open-loop strategies are

$$u_k^{\mathrm{OL}} = \tfrac{1}{2} \rho_k Q_k \varphi(t) \mathrm{e}^{rt} (1 - x_k^*), \quad k = 1, \dots, n \tag{15}$$

or in terms s_k^* and q_k,

$$u_k^{\mathrm{OL}} = \tfrac{1}{2} \rho_k q_k \varphi(t) \mathrm{e}^{rt} (m - s_k^*), \quad k = 1, \dots, n \tag{16}$$

where $\varphi(t)$ and x_k^* are as in (12). Comparison of the open- and closed-loop strategies reveals a close similarity. In fact we have

$$u_k^* = u_k^{\mathrm{OL}} \frac{(1 - x_k)}{(1 - x_k^*)} = u_k^{\mathrm{OL}} \frac{(m - s_k)}{(m - s_k^*)}. \tag{17}$$

Relationship (17) provides an important practical control rule; once we have derived the open-loop solution, which means solving the two-point boundary-value problem (TPBVP) (12), then the closed-loop solution is obtained simply by updating formula (14) by the real-time measurements $s_k(t)$. In strategic planning, formula (17) has the following advantage. The open-loop strategy can be derived in advance for the entire planning period. Then, during the implementation phase, when time is at a premium, and actual sales figures become available, the strategy can be rapidly modified to generate a closed-loop strategy for a particular time-period. Formula (17) can also be regarded as the rule that adapts prediction (16) to current market conditions. Note that, in the particular case when the market is saturated (i.e. total industry sales are fixed and equal to m) and $n = 2$, this result coincides with that of Fruchter and Kalish.[16]

3.2. Analysis of the closed-loop solution

The closed-loop advertising strategy of our model is an *own-brand sales response*. We shall examine the behaviour of the equilibrium closed-loop advertising strategy (14). We learn that equilibrium advertising, u_k^{*2}, is proportional to the square of potential sales, $(m - s_k)^2$. If the sales of the firm increase, then its potential sales decrease. In Note 2 in Appendix I we show that $\varphi(t)\mathrm{e}^{rt}$ decreases when sales increase, and therefore (14) reveals that the advertising expenditure of the firm targeted to attract potential sales should decrease if the sales of the firm increase.

Proposition 2

In a competitive growing dynamic market, firms should decrease their offensive advertising expenditures as their own sales increase or, equivalently, as their potential sales decreases.

Another property of the strategy under examination is that the time-variant coefficient, $\varphi(t)\mathrm{e}^{rt}$, is the same for all firms.

Proposition 3

The ratio of the advertising expenditures of any two firms in competition is time-invariant, and depends only on the ratios of their advertising effectiveness, gross profit margins and on their potential sales.

4. A TOTAL SALES FEEDBACK STRATEGY

We now wish to develop a time-invariant feedback strategy for the differential game associated with (9) or (11) which depends on the size of the market.

Theorem 2

Consider the differential game associated with (9) (*or* (11)). *Let*

$$z = \sum_{k=1}^{n} x_k = \frac{1}{m} \sum_{k=1}^{n} s_k \tag{18}$$

and suppose $f(z)$ *satisfies the following backward differential equation*:

$$nrf(z) = 1 + \frac{2n-1}{2}\left[f(z)f'(z)(1-z)^2 - f^2(z)(1-z)\right] \sum_{i=1}^{n} \rho_i^2 Q_i, \quad \lim_{t\to\infty} f(z)\mathrm{e}^{-rt} = 0 \tag{19}$$

Let $\hat{u}_k^*$, $k = 1, \ldots, n$, be a function of z, such that,

$$\hat{u}_k^* = \tfrac{1}{2}\rho_k Q_k f(z)(1-z) = \tfrac{1}{2}\rho_k q_k f\left(\frac{1}{m}\sum_{k=1}^{n} s_k\right)\left(m - \sum_{k=1}^{n} s_k\right), \quad k = 1, \ldots, n \tag{20}$$

Then $(\hat{u}_1^*, \ldots, \hat{u}_n^*)$ forms a time-invariant feedback Nash equilibrium strategy for the above differential game.

Proof. See Appendix III. □

Remark 2

If $r = 0$, (19) can be integrated analytically. We thus obtain

$$f(z) = \frac{2}{[(2n-1)(1-z)\sum_{i=1}^{n}\rho_i^2 Q_i]^{1/2}} \tag{21}$$

As a result (20) becomes

$$\hat{u}_k^* = \frac{\rho_k Q_k (1-z)^{1/2}}{[(2n-1)\sum_{i=1}^{n}\rho_i^2 Q_i]^{1/2}} \quad \text{or} \quad \hat{u}_k^{*2} = \frac{\rho_k^2 q_k^2 (m - \sum_{k=1}^{n} s_k)}{(2n-1)\sum_{k=1}^{n}\rho_k^2 q_k} \tag{22}$$

Considering (22) we learn that for a zero discount rate the equilibrium advertising strategy, $\hat{u}_k^{*2}$, decreases linearly with the actual total industry sales.

If $r > 0$, (19) can be integrated numerically. Simulation shows that the equilibrium advertising strategy decreases nonlinearly with the actual total industry sales.

Proposition 4

In a competitive growing dynamic market, if the firms only are aware of total market sales, considering only the effects of offensive strategies, then they should decrease their advertising expenditures as the total industry sales increase.

6. CONCLUSIONS AND IMPLICATIONS FOR FUTURE RESEARCH

Much has been published on competitive advertising using the Lanchester model to capture the dynamics of sales. Until now, the research has produced dynamic market share response models that are restricted to a saturated market, which is not always appropriate. The present study removes this assumption from the Lanchester model by defining a dynamic sales response model. The new model broadens the Lanchester model to take into account a market where the total industry has the potential to grow as a result of advertising activities, thus filling a major gap in our ability to represent conditions that commonly occur in the marketplace.

While the current study greatly broadens the market conditions that we are able to model, it also points to further gaps. There is a need for a model that incorporates external disturbances in the market whose influence runs counter to the objectives and marketing efforts of the firms. Market disturbances due to such factors will explain how total industry sales may decline even in the face of aggressive marketing—as may indeed occur in the marketplace. Another logical expansion of this study would be to find a defensive (retention) strategy that could operate simultaneously with the offensive (acquisition) strategy.

APPENDIX I: DERIVATION OF THE OPEN- AND CLOSED-LOOP STRATEGIES

I.1. The algorithm of constructing the closed-loop strategies

Considering the differential game associated with (11), we find that the current value Hamiltonian of player k is given by, cf. Kamien and Schwartz[17],

$$H_k = H_k(x_k, u_1, \dots, u_n, \lambda_k, t) = Q_k x_k - u_k^2 + \lambda_k \dot{x}_k \tag{23}$$

where λ_k is the adjoint variable or, in our context, the marginal value of having the constraint relaxed by one unit. The necessary optimality conditions are given by

$$\partial H_k / \partial u_k = [-2u_k + \rho_k \lambda_k (1 - x_k)] = 0, \quad \text{or } u_k = \tfrac{1}{2} \rho_k \lambda_k (1 - x_k) \tag{24}$$

and the adjoint equation is

$$\dot{\lambda} = r\lambda_k - \partial H_k / \partial x_k = r\lambda_k - Q_k + \tfrac{1}{2} \lambda_k \sum_{j=1}^{n} \rho_j u_j, \quad \lim_{t \to \infty} \lambda_k(t) \mathrm{e}^{-rt} = 0 \tag{25}$$

Substituting (24) into (11) and (25), the following two-point boundary-value problem (TPBVP) is obtained:

$$\dot{x}_k = \frac{1}{2} \left[\rho_k^2 \lambda_k (1 - x_k) - x_k \sum_{j=1}^{n} \rho_j^2 \lambda_j (1 - x_j) \right], \quad x_k(0) = x_k^0 \tag{26}$$

$$\dot{\lambda}_k = r\lambda_k - Q_k + \frac{1}{2} \lambda_k \sum_{j=1}^{n} \rho_j^2 \lambda_j (1 - x_j), \quad \lim_{t \to \infty} \lambda_k \mathrm{e}^{-rt} = 0 \tag{27}$$

Note 1

From the phase-plane portrait of equations (26) *and* (27), *described in Figure* 1, *we see that the trajectory that satisfies the necessary conditions belongs to region I.*[‡]

To find the values for u_k that produce a stationary value for Π_k, we must solve the TPBVP (26) and (27)). We will use the notation x_k^* for the value of x_k that solves (26) and (27). Let

$$\lambda_k(t) = Q_k \varphi(t) \mathrm{e}^{rt}. \tag{28}$$

In terms of the new variable $\varphi(t)$, (27) becomes[§]

$$\dot{\varphi}(t) = \frac{1}{2} \sum_{j=1}^{n} \rho_j^2 Q_j (1 - x_j^*(t)) \varphi^2(t) \mathrm{e}^{rt} - \mathrm{e}^{rt}, \quad \varphi(\infty) = 0. \tag{29}$$

I.2. The role of (29)

Equation (29) replaces the set of n equations in (27). In this way we considerably reduce the complexity of the computation required to solve the TPBVP ((26) and (27)). In conclusion, the

‡ Indeed plugging $x_k = 0$ (or $s_k = 0$, as $x_k = s_k/m$) into (26) we obtain $\dot{x}_k > 0$ (or $\dot{s}_k > 0$), and $\lambda_k = 0$ into (27) we obtain $\dot{\lambda}_k < 0$; therefore the trajectory which satisfies (26) and (27) should belong to region I.

§ Equation (29) follows immediately by substituting (28) and its derivative with respect to t into (27). Equivalently, considering (28) and (29) one can obtain back (27).

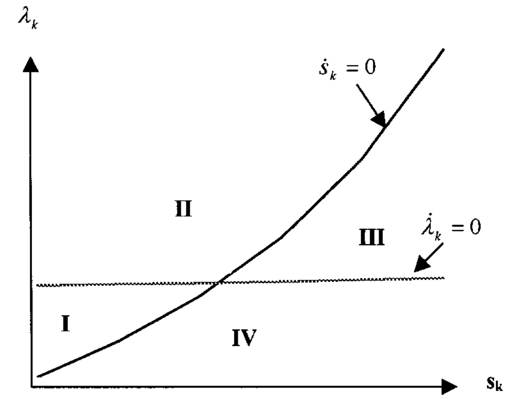

Figure 1. Phase-plane portrait of equations (26) and (27)

TPBVP ((26) and (27)) of $2n$ equations is equivalent to the following TPBVP of $n+1$ equations,

$$\dot{x}_k^*(t) = \frac{1}{2}\left[\rho_k^2 Q_k \varphi(t)(1 - x_k^*(t)) - x_k^*(t)\sum_{j=1}^{n} \rho_j^2 Q_j \varphi(t)(1 - x_j^*(t))\right]\mathrm{e}^{rt}, \quad x_k^*(0) = x_k^0, \quad k = 1, \ldots, n$$

$$\dot{\varphi}(t) = \frac{1}{2}\sum_{j=1}^{n} \rho_j^2 Q_j (1 - x_j^*(t))\varphi^2(t)\mathrm{e}^{rt} - \mathrm{e}^{-rt}, \quad \varphi(\infty) = 0 \tag{30}$$

I.3. The open-loop strategy

Let $\varphi(t)$ and x_k^* be as in (30). Then the open-loop strategies are given by

$$u_k^{\mathrm{OL}} = \tfrac{1}{2}\rho_k Q_k \varphi(t)\mathrm{e}^{rt}(1 - x_k^*), \quad k = 1, \ldots, n \tag{31}$$

or in terms s_k and q_k,

$$u_k^{\mathrm{OL}} = \tfrac{1}{2}\rho_k q_k \varphi(t)\mathrm{e}^{rt}(m - s_k^*), \quad k = 1, \ldots, n \tag{32}$$

I.4. The closed-loop strategy

Let $\varphi(t)$ be as in (30) and let x_k be as in (11). Then the closed-loop strategies are given by

$$u_k^* = \tfrac{1}{2}\rho_k Q_k \varphi(t)\mathrm{e}^{rt}(1 - x_k), \quad k = 1, \ldots, n \tag{33}$$

or

$$u_k^* = \tfrac{1}{2}\rho_k q_k \varphi(t)\mathrm{e}^{rt}(m - s_k), \quad k = 1, \ldots, n \tag{34}$$

where s_k is as in (9).

Note 2

Considering Note 1, (28) *and* (34), *if s_k increases then $\varphi(t)\mathrm{e}^{rt}$ decreases, and consequently u_k^* decreases.*

APPENDIX II: PROOF OF THEOREM 1

Consider the zero sum

$$0 = -Q_k\varphi(t)x_k\Big|_0^\infty + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}(Q_k\varphi(t)x_k)\,\mathrm{d}t \tag{35}$$

where $\varphi(t)$ is as in (29), and x_k as in (3). From (35) we obtain

$$0 = Q_k\varphi(0)x_k^0 + \int_0^\infty [Q_k\dot{\varphi}(t)x_k + Q_k\varphi(t)\dot{x}_k]\,\mathrm{d}t \tag{36}$$

Using equation (11) for $\dot{x}_k$, we obtain

$$0 = Q_k\varphi(0)x_k^0 + \int_0^\infty \left[Q_k\dot{\varphi}(t)x_k + Q_k\varphi(t)\left(\rho_k u_k - x_k\sum_{j=1}^{n}\rho_j u_j\right)\right]\mathrm{d}t \tag{37}$$

Considering equation (29) for $\dot{\varphi}$, we obtain

$$0 = Q_k\varphi(0)x_k^0 + \int_0^\infty\left\{\frac{1}{2}Q_k\left[\sum_{j=1}^{n}\rho_j^2Q_j(1-x_j^*)\varphi^2(t)\mathrm{e}^{rt} - \mathrm{e}^{-rt}\right]x_k + Q_k\varphi(t)\left(q_k u_k - x_k\sum_{j=1}^{n}q_j u_j\right)\right\}\mathrm{d}t \tag{38}$$

Considering (31)–(34), equation (36) becomes

$$0 = Q_k\varphi(0)x_k^0 + \int_0^\infty [2(u_k - u_k^{\mathrm{OL}})u_k^* + Q_k\rho_k\varphi(t)\mathrm{e}^{rt}u_k^{\mathrm{OL}}$$
$$+ 2\sum_{\substack{j=1\\ j\neq k}}^{n}\rho_j(u_j - u_j^{\mathrm{OL}})(u_k^*/\rho_k) - Q_k\varphi(t)\mathrm{e}^{rt}\sum_{j=1}^{n}\rho_j(u_j - u_j^{\mathrm{OL}}) - Q_kx_k\Bigg]\mathrm{e}^{-rt}\,\mathrm{d}t \tag{39}$$

Adding the zero sum (39) to $\Pi_k = \Pi_k(u_1, \ldots, u_k)$ in (11), we obtain

$$\Pi_k(u, \ldots, u_n) = Q_k\varphi(0)x_k^0 + \int_0^\infty\Bigg[-(u_k - u_k^*)^2 + {u_k^*}^2 - 2u_k^{\mathrm{OL}}u_k^* + Q_k\rho_k\varphi(t)\mathrm{e}^{rt}u_k^{\mathrm{OL}}$$
$$+ 2\sum_{\substack{j=1\\ j\neq k}}^{n}\rho_j(u_j - u_j^{\mathrm{OL}})(u_k^*/\rho_k) - Q_k\varphi(t)\mathrm{e}^{rt}\sum_{\substack{j=1\\ j\neq k}}^{n}\rho_j(u_j - u_j^{\mathrm{OL}})\Bigg]\mathrm{e}^{-rt}\,\mathrm{d}t \tag{40}$$

Particularly,

$$\Pi_k(u_1^*, \ldots, u_n^*) = Q_k\varphi(0)s_k^0 + \int_0^\infty\Bigg[({u_k^*}^2 - 2u_k^{\mathrm{OL}}u_k^* + Q_k\rho_k\varphi(t)\mathrm{e}^{rt}u_k^{\mathrm{OL}}$$
$$+ 2\sum_{\substack{j=1\\ j\neq k}}^{n}\rho_j(u_j^* - u_j^{\mathrm{OL}})(u_k^*/\rho_k) - Q_k\varphi(t)\mathrm{e}^{rt}\sum_{\substack{j=1\\ j\neq k}}^{n}\rho_j(u_j^* - u_j^{\mathrm{OL}})\Bigg]\mathrm{e}^{-rt}\,\mathrm{d}t \tag{41}$$

Considering (40) and (41), we have

$$\Pi_k(u_1^*, \dots, u_{k-1}^*, u_k, u_{k+1}^*, \dots, u_n^*) = Q_{k,}\varphi(0)x_k^0 + \int_0^\infty -(u_k - u_k^*)^2 + u_k^{*2} - 2u_k^{\mathrm{OL}}u_k^*$$

$$+ Q_k\rho_k\varphi(t)\mathrm{e}^{rt}u_k^{\mathrm{OL}} + 2\left[\sum_{\substack{j=1\\ j\neq 1}}^{n} \rho_j(u_j^* - u_j^{\mathrm{OL}})(u_k^*/\rho_k)\right.$$

$$\left. - Q_k\varphi(t)\mathrm{e}^{rt}\sum_{\substack{j-1\\ j\neq k}}^{n} \rho_j(u_j^* - u_j^{\mathrm{OL}})\right]\mathrm{e}^{-rt}\,\mathrm{d}t$$

$$= \Pi_k(u_1^*, \dots, u_n^*) - \int_0^\infty (u_k - u_k^*)^2\mathrm{e}^{-rt}\,\mathrm{d}t \leqslant \Pi_k(u_1^*, \dots, u_n^*) \tag{42}$$

for every u_k. This completes the proof of our theorem. □

APPENDIX III: PROOF OF THEOREM 2

The current value Hamiltonians of this problem are

$$H_k = Q_k x_k - u_k^2 + \sum_{i=1}^{n} \lambda_k^i\left(\rho_i u_i - x_i \sum_{j=1}^{n} \rho_j u_j\right), \quad k = 1, 2, \dots, n \tag{43}$$

and λ_k^i, i, $k = 1, 2, \dots, n$ are the co-state variables of this problem. Setting $\partial H_k/\partial u_k = 0$, $k = 1, 2, \dots, n$, one finds

$$\hat{u}_k = \frac{1}{2}\rho_k\left(\lambda_k^k - \sum_{i=1}^{n} \lambda_k^i x_i\right) \tag{44}$$

Making the identifications $\lambda_k^i = V_{x_i}^k = \partial V^k/\partial x_i$, and substituting control functions (44) into (43), one obtains the Hamilton–Jacobi–Bellman (HJB) equations

$$m_k + rV^k = q_k x_k - \hat{u}_k^2 + \sum_{i=1}^{n} V_{x_i}^k\left(\rho_i \hat{u}_i - x_i \sum_{j=1}^{n} \rho_j \hat{u}_j\right), \quad k = 1, 2, \dots, n \tag{45}$$

where

$$\hat{u}_k = \frac{1}{2}\rho_k\left(V_{x_{ki}}^k - \sum_{i=1}^{n} V_{x_i}^k x_i\right) \tag{46}$$

and m_k is an arbitrary real constant as above. The equations in (45) constitute a system of n simultaneous non-linear partial differential equations of the n unknown functions $V^k(x_1, x_2, \dots, x_n)$, $k = 1, 2, \dots, n$. Since this is a *very difficult problem* we want to find a special solution for which $V_{x_j}^k = V_{x_k}^k$, $k, j = 1, \dots, n$, and in this way simplify (45) and (43). More exactly, we want to find a function $F(z)$, where

$$z = x_1 + x_2 + \cdots + x_n \tag{47}$$

such that (45) will admit the solutions

$$V^k(t, x_1, \ldots, x_n) = Q_k F(z) \tag{48}$$

It follows that

$$V^k_{x_1} = V^k_{x_2} = \cdots = V^k_{x_n} = Q_k F'(z) = Q_k f(z), \quad k = 1,2, \ldots, n \tag{49}$$

Considering (47)–(49), equation (45) will become, after manipulation

$$\sum_{i=1}^{n} \frac{m_i}{Q_i} + nrF(z) = z + \frac{2n-1}{4} f^2(z)(1-z)^2 \sum_{i=1}^{n} \rho_i^2 Q_i \tag{50}$$

If $r = 0$, then $m_i = Q_i/n$ (to see this substitute $z = 1$). If we take the derivative of (50) with respect to z, we obtain

$$nrf(z) = 1 + \frac{2n-1}{2} [f(z)f'(z)(1-z)^2 - f^2(z)(1-z)] \sum_{i=1}^{n} \rho_i^2 Q_i \tag{51}$$

From terminal conditions we must have,

$$\lim_{t \to \infty} f(z) \mathrm{e}^{-rt} = 0 \tag{52}$$

By solving (51) and (52) we can obtain $f(z)$ and therefore

$$\hat{u}_k(z) = \tfrac{1}{2} \rho_k Q_k f(z)(1-z) \tag{53}$$

or in terms of s_k we obtain

$$\hat{u}_k(s_1, \ldots, s_n) = \frac{1}{2} \rho_k q_k f\left(\frac{1}{m} \sum_{k=1}^{n} s_k\right)\left(m - \sum_{k=1}^{n} s_k\right) \tag{54}$$

The strategy in (54) forms a time-invariant feedback Nash equilibrium strategy since it satisfies HJB equation. □

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